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# Fuzzy Linear Programming and its Application for a Constructive Proof of a Fuzzy Version of Farkas Lemma

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**Abstract** The main aim of this paper is to deal with a fuzzy version of Farkas lemma involving trapezoidal fuzzy numbers. In turns to that the fuzzy linear programming and duality theory on these problems can be used to provide a constructive proof for Farkas lemma.

**Keywords** Farkas Lemma, Fuzzy Linear Programming, Duality, Ranking Functions.

## 1 Introduction

Farkas' lemma is a key ingredient to establish the Karush-Kuhn-Tucker optimality conditions that are both necessary and sufficient for linear programming problems. The result deals with two linear systems of equations and inequalities, exactly one of which has a solution in any instance. But sometimes in real situations there is an ambiguity in some parameters of the system. Frankly, it is possible that some of the parameters of the linear systems be fuzzy numbers. Therefore, the studies of these systems are important in the literature. In this work, we focus on Farkas' lemma and define a fuzzy version of it. Then we give a constructive proof of the fuzzy version of Farkas lemma by using the fuzzy linear programming and duality theory on these problems established by Mahdavi-Amiri and Nasseri [1]. This paper is organized as follows: In Section 2, we review some preliminaries which are needed in next sections. We define a fuzzy linear programming in Section 3and give some duality results on these problems. In Section 4, we give a fuzzy version of Farkas' lemma and also give a constructive proof for fuzzy Farkas lemma by using fuzzy linear programming problems and duality theory.

## 2 Preliminaries

### 2.1 Definitions and Notations

We review the fundamental notions of fuzzy set theory.

**Definition 1.** A convex fuzzy set $\tilde{A}$on $\mathbb{R}$ is a fuzzy number if the following conditions hold:
(a) Its membership function is piece-wisely continuous.
(b)There exist three intervals $[a,b]$, $[b,c]$and $[c,d]$such that $\mu_{\tilde{A}}$is increasing on $[a,b]$, equal to 1 on $[b,c]$, decreasing on $[c,d]$ and equal to 0 elsewhere.

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**Definition 2.** Let $\tilde{A} = (a^L, a^U, \alpha, \beta)$ denote the trapezoidal fuzzy number, where $(a^L - \alpha, a^U - \beta)$ is the support of $\tilde{A}$ and $[a^L, a^U]$ its core.
**Remark 1.** We denote the set of all trapezoidal fuzzy numbers by $\mathcal{F}(\mathbb{R})$.

Now we define the concept of arithmetic on trapezoidal fuzzy numbers which is useful in throughout the paper. Let $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$ be two trapezoidal fuzzy numbers. Define,

$$x \geq 0, x \in \mathbb{R};\ x\tilde{a} = (xa^L, xa^U, x\alpha, x\beta)$$
$$x < 0, x \in \mathbb{R};\ x\tilde{a} = (xa^U, xa^L, -x\beta, -x\alpha)$$
$$\tilde{a} + \tilde{b} = (a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \theta).$$
$$\tilde{a} - \tilde{b} = (a^L - b^U, a^U - b^L, \alpha + \theta, \beta + \gamma).$$

### 2.2 Ranking functions

One convenient approach for solving the fuzzy linear programming problems is based on the concept of comparison of fuzzy numbers through the use of ranking functions [1, 2, 3]. An effective approach for ordering the elements of $\mathcal{F}(\mathbb{R})$ is to define a ranking function $\mathcal{R}: \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ which maps each fuzzy number into the real line, where a natural order exists.

We define orders on $\mathcal{F}(\mathbb{R})$ by:

$$\tilde{a} \succcurlyeq \tilde{b} \text{ if and only if } \mathcal{R}(\tilde{a}) \geq \mathcal{R}(\tilde{b}) \quad (1)$$
$$\tilde{a} \succ \tilde{b} \text{ if and only if } \mathcal{R}(\tilde{a}) > R(\tilde{b}) \quad (2)$$
$$\tilde{a} \simeq \tilde{b} \text{ if and only if } \mathcal{R}(\tilde{a}) = \mathcal{R}(\tilde{b}) \quad (3)$$

where $\tilde{a}$ and $\tilde{b}$ are in $\mathcal{F}(\mathbb{R})$. Also we write $\tilde{a} \preccurlyeq \tilde{b}$ if and only if $\tilde{b} \succcurlyeq \tilde{a}$.

We restrict our attention to linear ranking functions.
**Remark 2.** For any trapezoidal fuzzy number $\tilde{a}$, the relation $\tilde{a} \succcurlyeq \tilde{0}$ holds, if there exist $\varepsilon \geq 0$ and $\alpha > 0$ such that $\tilde{a} \succcurlyeq (-\varepsilon, \varepsilon, \alpha, \alpha)$. We realize that $\mathcal{R}(-\varepsilon, \varepsilon, \alpha, \alpha) = 0$ (we also consider $\tilde{a} \simeq \tilde{0}$ if and only if $\mathcal{R}(\tilde{a}) = 0$). Thus, without loss of generality, throughout the paper we let $\tilde{0} = (0, 0, 0, 0)$ as the zero trapezoidal fuzzy number.
The following lemma is now immediately in hand [2].
**Lemma 1.** Let $\mathcal{R}$ be any linear ranking function. Then,
(i) $\tilde{a} \succcurlyeq \tilde{b}$ if and only if $\tilde{a} - \tilde{b} \succcurlyeq \tilde{0}$ if and only if $-\tilde{b} \succcurlyeq -\tilde{a}$
(ii) If $\tilde{a} \succcurlyeq \tilde{b}$ and $\tilde{c} \succcurlyeq \tilde{d}$, then $\tilde{a} + \tilde{c} \succcurlyeq \tilde{b} + \tilde{d}$.

We consider the linear ranking functions on $\mathcal{F}(\mathbb{R})$ as:

$$\mathcal{R}(\tilde{a}) = c_L a^L + c_U a^U + c_\alpha \alpha + c_\beta \beta \quad (4)$$

where $\tilde{a} = (a^L, a^U, \alpha, \beta)$, and $c_L, c_U, c_\alpha, c_\beta$ are constants, at least one of which is non-zero. A special version of the above linear ranking function was first proposed by Yager [2, 3] as follows:

$$\mathcal{R}(\tilde{a}) = \frac{1}{2}\int_0^1 (\inf \tilde{a}_\lambda + \sup \tilde{a}_\lambda) d\lambda \tag{5}$$

which reduces to

$$\mathcal{R}(\tilde{a}) = \frac{a^L + a^U}{2} + \frac{1}{4}(\beta - \alpha) \tag{6}$$

Then, for trapezoidal fuzzy numbers $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$, we have $\tilde{a} \succcurlyeq \tilde{b}$ if and only if $a^L + a^U + \frac{1}{2}(\beta - \alpha) \geq b^L + b^U + \frac{1}{2}(\theta - \gamma)$ (7)

## 3 Fuzzy linear programming problems

Authors who use ranking functions for comparison of fuzzy numbers usually define a crisp model which is equivalent to the fuzzy linear programming problem and then use the optimal solution of this model as the optimal solution of fuzzy linear programming problem [1, 2].We now define fuzzy linear programming problems and the corresponding crisp models.

### 3.1 Formulation of the fuzzy linear programming problem

**Definition 3.** A Fuzzy Linear Programming (FLP) problem is defined as follows:

$$\begin{array}{ll} \text{Max} & \tilde{z} \simeq \tilde{c}x, \\ \text{s.t.} & \tilde{A}x \preccurlyeq \tilde{b}, \\ & x \geq 0. \end{array} \tag{8}$$

where $\tilde{A} = (\tilde{a}_{ij})_{m\times n}, \tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_n), \tilde{b} = \left(\tilde{b}_1, \dots, \tilde{b}_m\right)^T$ and $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_j \in \mathcal{F}(\mathbb{R})$ for $i = 1, \dots, m,\ j = 1, \dots, n$.

**Definition 4.** Any $x$ which satisfies the set of constraints of FLP is called a feasible solution. Let $Q$ be the set of all feasible solutions of FLP. We say that $x^0 \in Q$is an optimal feasible solution for FLP if $\tilde{c}x \leq \tilde{c}x^0$ for all $x \in Q$.

**Definition 5.** We say that the real number a corresponds to the fuzzy number $\tilde{a}$, with respect to a given linear ranking function $\mathcal{R}$, if $a = \mathcal{R}(\tilde{a})$.

The following theorem shows that any FLP can be reduced to a linear programming problem (see in [2]).

$$\begin{array}{ll} \text{Max} & \tilde{z} \simeq \tilde{c}x, \\ \text{s.t.} & \tilde{A}x \preccurlyeq \tilde{b}, \\ & x \geq 0. \end{array} \tag{9}$$

where $a_{ij}, b_i, c_j$ are real numbers corresponding to the fuzzy numbers $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_j$ with respect to a given linear ranking function $\mathcal{R}$, respectively.

**Remark 3.** The above theorem shows that the sets of all feasible solutions of FLP problem and LP problem are the same. Also if $\bar{x}$ is an optimal feasible solution for FLP problem, then $\bar{x}$ is an optimal feasible solution for LP problem.

**Corollary 1.** If LP problem does not have a solution then FLP problem does not have a solution either.

### 3.1 Optimality Conditions

Consider the FLP problem (in the standard form),

$$\begin{array}{ll} \text{Max} & \tilde{z} \simeq \tilde{c}x, \\ \text{s.t.} & \tilde{A}x \simeq \tilde{b}, \\ & x \geq 0. \end{array} \tag{10}$$

where the parameters of the problem are as defined in (8).
The following theorem characterizes optimal solutions. The result corresponds to the so-called nondegenerate problems, where all basic variables corresponding to every basis B are nonzero (and hence positive).

**Theorem 1.** Assume the FLP problem is nondegenerate. A basic feasible solution $x_B = B^{-1}b, x_N = 0$ is optimal to (10) if and only if $\tilde{z}_j \succcurlyeq \tilde{c}_j$ for all $1 \leq j \leq n$.

## 4 Duality in fuzzy linear programming

Similar to the duality theory in linear programming (see for example [4]), for every FLP problem, there is an associated problem which satisfies some important properties. We shall call this related FLP problem the DFLP problem [1].

### 4.1 Formulation of the dual problem. For the FLP problem

$$\begin{array}{ll} \text{Max} & \tilde{z} \simeq \tilde{c}x, \\ \text{s.t.} & \tilde{A}x \preccurlyeq \tilde{b}, \\ & x \geq 0. \end{array} \tag{11}$$

define the Dual Fuzzy Linear Programming (DFLP) problem as:

$$\begin{array}{ll} \text{Min} & \tilde{u} \simeq \omega\tilde{b}, \\ \text{s.t.} & \omega\tilde{A} \succcurlyeq \tilde{c}, \\ & \omega \geq 0. \end{array} \tag{12}$$

Note that there is exactly one dual variable (of the form $\geq 0$) for each FLP problem constraint of the form $\preccurlyeq$ and exactly one dual constraint (of the form $\succcurlyeq$) for each variable of the form $\geq$ in FLP problem.

### 4.2 The relationships between FLP and DFLP problems

We shall give here the relationships between the fuzzy linear programming problem and its corresponding duality and omit the proofs (for more details see [1]).
**Lemma 2.** Dual of DFLP problem is FLP problem.

**Remark 4.** Lemma 4 indicates that the duality results can be applied to any of the primal or dual problems posed as the primal problem.
**Theorem 2.** (The Weak Duality Property.) If $x_0$ and $w_0$ are feasible solutions to FLP problem and DFLP problem, respectively, then $\tilde{c}x_0 \preccurlyeq w_0\tilde{b}$.
**Remark 5.** The value of the ranking function for the fuzzy value of the objective function at any feasible solution to FLP problem is always lower than or equal to the value of the ranking function for the fuzzy value of the objective function for any feasible solution to DFLP problem.
The following corollaries are immediate consequences of Theorem 2.
**Corollary 2.** If $x_0$ and $w_0$ are feasible solutions to FLP problem and DFLP problem, respectively, and $\tilde{c}x_0 \simeq w_0\tilde{b}$, then $x_0$ and $w_0$ are optimal solutions to their respective problems.
The following corollary relates unboundness of one problem to infeasibility of the other. We use the definition below.
**Definition 6.** We say FLP problem (or DFLP problem) is unbounded if feasible solutions exist with arbitrary large (or small) ranking function for the fuzzy objective value.
**Corollary 3.** If either problem is unbounded, then the other problem has no feasible solution.
We now state the main duality result.
**Theorem 3.** (Strong Duality) If any one of the FLP problem or DFLP problem has an optimal solution, then both problems have optimal solutions and the two optimal values of ranking functions for the fuzzy objective values are equal.
Using the results of the lemmas, corollaries, and remarks above we obtain the following important duality result.
**Theorem 4.** (Fundamental Theorem of Duality.) For any FLP problem and its corresponding DFLP problem, exactly one of the following statements is true.
(1) Both have optimal solutions $x_*$ and $w_*$ with $\tilde{c}x_* \simeq w_*\tilde{b}$.
(2) One problem is unbounded and the other is infeasible.
(3) Both problems are infeasible.

## 5 A Fuzzy Version of Farkas Lemma

Farkas lemma is a key ingredient to establish the Karush-Kuhn-Tucker optimality conditions that are both necessary and sufficient for linear programming problems [4]. Here we give a fuzzy version of this result.

**Lemma 5. (Fuzzy Farkas' Lemma)**
Consider the following fuzzy linear systems,

$$\tilde{A}x \succcurlyeq \tilde{0}, \tilde{c}x \prec \tilde{0} \tag{13}$$
$$y\tilde{A} \simeq \tilde{c}, y \geq 0 \tag{14}$$

**Proof.** It is enough to show that if there exists an $x$ with $\tilde{A}x \succcurlyeq \tilde{0}$ and $\tilde{c}x \prec \tilde{0}$, then there is not any $y \geq 0$ with $y\tilde{A} \simeq \tilde{c}$. Conversely, if there exists no $x$ with $\tilde{A}x \succcurlyeq \tilde{0}$ and $\tilde{c}x \prec \tilde{0}$, then there exists a $y \geq 0$ such that $y\tilde{A} \simeq \tilde{c}$. First, suppose that System (13) has a solution $x$. If System

(14) also has a solution $y$, then by using Lemma 1 we have $\tilde{c}x \simeq y\tilde{A}x \succcurlyeq \tilde{0}$, since $y \geq 0$ and $\tilde{A}x \succcurlyeq \tilde{0}$. This contradicts $\tilde{c}x \prec \tilde{0}$; therefore, System (14) cannot have a solution.
Next, suppose that System (13) does not have a solution. Consider the following problems:

$$\begin{array}{ll} \text{Min} & \tilde{c}x, \\ \text{s.t.} & \tilde{A}x \succcurlyeq \tilde{0}. \end{array} \tag{15}$$

$$\begin{array}{ll} \text{Max} & y\tilde{0} \simeq \tilde{0}, \\ \text{s.t.} & y\tilde{A} \simeq \tilde{c}, \\ & y \geq 0. \end{array} \tag{16}$$

It is obvious that Problem (15) is feasible and $\tilde{d} = 0$ is the optimal solution for Problem (15) (the optimal fuzzy value of the objective function is zero). System (13) has not any solution. Now by using Theorem 3 (Strong Duality Theorem), Problem (16) is feasible and its optimal fuzzy value of the objective function is zero. Now if $\tilde{p}$ be every feasible solution for Problem (16) then $\tilde{p}$ is a solution for System (14). The proof is complete.

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**References**

1. Mahdavi-Amiri, N., Nasseri, S. H., (2006). Duality in fuzzy number linear programming by use of a certain linear ranking function. Applied Mathematics and Computation, 180, 206–216.
2. Maleki, H. R., Tata, M., Mashinchi, M., (2000). Linear programming with fuzzy variables. Fuzzy Sets and Systems, 109, 21–33.
3. Mahdavi-Amiri, N., Nasseri, S. H., (2007). Duality results and a dual simplex method for linear programming problems with trapezoidal fuzzy variables. Fuzzy Sets and Systems, 158, 1961–1978.
4. Bazaraa, M. S., Jarvis, J. J., Sherali, H. D., (1990). Linear Programming and Network Flows. John Wiley, New York, Second Edition.