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Solving fully fuzzy Linear Programming Problem using Breaking Points

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Abstract In this paper we have investigated a fuzzy linear programming problem with fuzzy quantities which are LR triangular fuzzy numbers. The given linear programming problem is rearranged according to the satisfactory level of constraints using breaking point method. By considering the constraints, the arranged problem has been investigated for all optimal solutions connected with satisfactory level of quantities on all intervals that constituted from breaking points of the constraints. Optimal solution could be obtained on the constructed $[\gamma_{p-1}, \gamma_p]$ intervals for p=1,2… Here the fuzzy primal simplex algorithm is designed based on revised TSAO's ranking method.

Keywords Triangular Fuzzy Number, Ranking Fuzzy Number, Revised Tsao's Method ,Fuzzy Linear Programming Problem, Breaking Point, Fuzzy Primal Simplex Algorithm.

1 Introduction

Fuzzy linear programming was first formulated by Zimmermann [1]. Recently, these problems are considered in several kinds, that is, it is possible that some coefficients of the problem in the objective function, technical coefficients, the right-hand side coefficients or decision making variables be fuzzy numbers [2], [3], [4], [5], [6], [7],[8]. In this work, we focus on the linear programming problems with LR triangular fuzzy numbers.

Here, we first explain the concept of the breaking points. In this paper, we have investigated a fuzzy linear programming problem with fuzzy quantities which are bounded fuzzy triangular numbers and the given problem is rearranged according to the satisfactory level. Considering the constraints, the arranged problem has been investigated for all optimal solution connected with satisfactory level for quantities on all intervals that constituted from breaking points of the objective cost from zero to maximum satisfactory level. We proposed a technique which finds all different optimal solutions on the $[\gamma_{p-1}, \gamma_p]$ intervals for p = 1, 2,…, for the fuzzy linear programming . Moreover, we describe basic feasible solution for the Fuzzy linear programming problems and state optimality conditions for these problems. Finally, we provide some important results for Fuzzy linear programming problems and we propose simplex algorithm for solving these problems. The breaking points of satisfactory level depending on constraints and changing optimal solutions are obtained successfully.

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2 Some notions on Fuzzy Sets

Definition 1.
If X is a collection of objects denoted generically by x, then a **fuzzy set** A in X is defined as a set of ordered pairs $A = \{(x,\mu_A(x)) \mid x \varepsilon X\}$, where, $\mu_A(x)$ is called the **membership function** (or MF for short) for the fuzzy set A. The MF maps each element of X to a membership grade (or membership value) between 0 and 1 (included).Obviously, the definition of a fuzzy set is a simple extension of the definition of a classical (crisp) set in which the characteristic function is permitted to have any values between 0 and 1. If the value of the membership function is restricted to either 0 or 1, then A is reduced to a classical set.

Definition 2.
The **support** of a fuzzy set A is the set of all points x in X such that $\mu_A(x) > 0$.
The **α − level (α −cut)** set of a fuzzy set A is a crisp subset of X and is denoted by
$A_\alpha = \{x\varepsilon X / \mu_A(x)) \geq \alpha\}$
A fuzzy set A in X is **convex** *if* $\mu_A(\lambda x + (1-\lambda)y) \geq \min\{(\mu_A(x), (\mu_A(y)\}$,for all
x ,y ε A and $\lambda \varepsilon [0,1]$

Definition 3.
A triangular fuzzy number $A(x)=(a_1, a_2, a_3)$ has in general a linear representation as shown in the following figure .

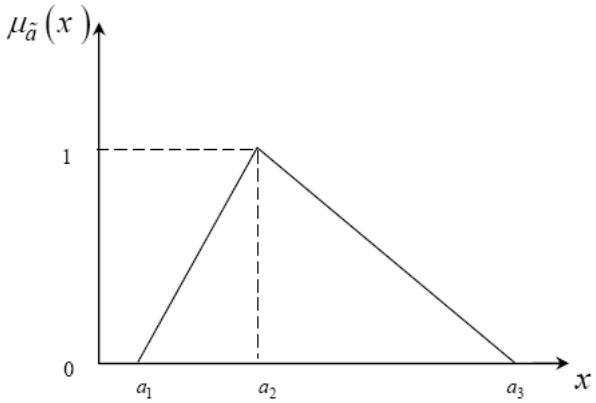

Fig .1 Triangular Fuzzy Number

Any triangular fuzzy number $A(x)=(a_1,a_2,a_3)$, can be written as follows:

$$A(x) = \begin{cases} (x-a_1)/(a_2-a_1)\ ,\ x \in [a_1\ ,\ a_2] \\ (a_3-x)/(a_3-a_2)\ ,x \in [a_2,a_3] \\ 0,\ x \leq a_1\ \&\ x \geq a_3 \end{cases}$$

The interval level of confidence is defined as:
$A^\alpha=[A_l^\alpha, A_r^\alpha]=[a_1+\alpha(a_2-a_1), a_3-\alpha(a_3-a_2)]$ where a_1,a_2,a_3, x ε R and α ε[0,1].

Definition 4.
If A=(a_1,a_2,a_3) and B=(b_1,b_2,b_3) with a_1,a_2,a_3,b_1,b_2,b_3 εR $^+$. The arithmetic operations are defined as follows:
A + B=(a_1+b_1,a_2+b_2,a_3+b_3) A - B=(a_1-b_1,a_2-b_2,a_3-b_3) A. B=(a_1b_1,a_2b_2,a_3b_3)
A/ B=($a_1/b_3,a_2/b_2,a_3/b_1$)

Definition 5.
A fuzzy eigen value λ is a fuzzy number which is a solution to the equation AX=λX where A is a n by n matrix containing fuzzy numbers and x is a non zero n by1 vector containing fuzzy numbers.

Remark
We consider $\tilde{0}$ = (0 , 0 , 0) as the zero triangular fuzzy number.

Definition 6.
The membership function f_A of A can be expressed as

$$f_A(x) = \begin{cases} f_A^L, a \le x \le b \\ w, b \le x \le c \\ f_A^R, c \le x \le d \\ 0, otherwise \end{cases}$$

where $f_A^L : [a,b] \to [0,w]$ $f_A^R : [c,d] \to [0,w]$. Since $f_A^L : [a,b] \to [0,w]$ is continuous and strictly
increasing, the inverse function of f_A^L exists. Similarly, since $f_A^R : [c,d] \to [0,w]$ is continuous and strictly decreasing, the inverse function of f_A^R also exists. The inverse functions of f_A^L and f_A^R can be denoted by g_A^L and g_A^R respectively. Since $f_A^L : [a,b] \to [0,w]$ is continuous and strictly increasing, $g_A^L : [0,w] \to [a,b]$ is continuous and strictly increasing,. Similarly, since $f_A^R : [c,d] \to [0,w]$ is continuous and strictly decreasing, $g_A^R : [0,w] \to [c,d]$ is continuous and strictly decreasing, the inverse function of f_A^R also exists. g_A^L and g_A^R are continuous on [0,w].They are integrable on [0,w] That is, both $\int_0^L g_A^L dx$ *and* $\int_0^R g_A^R dy$ exist.

Let $\bar{A}, \bar{B}$ are two fuzzy numbers of LR type:

$\bar{A}$ = (a, α, β)LR , $\bar{B}$ = (b, λ, δ)LR

Then

(a, α, β)LR ⊕ (b, λ, δ)LR = (a + b, α + λ, β + δ)LR

−(b, λ, δ) = (−b, δ, λ)LR

(a, α, β)LR − (b, λ, δ)LR = (a − b, α + δ, β + λ)LR

3 The Revised Method of Ranking Fuzzy Numbers with an Area Between the Centroid and Original points.

Ranking fuzzy number with an area between the centroid and the original points using the revised version of Chu and Tsao. Reference [9] relinquished the area proposed by Chu and Tsao but instead considered centroid-point. However, the revised version seems escalate in the complexity of computation. Moreover, validation of the revised method was solely banked on several hypothetical examples and far too little attention has been paid to test it into real case study. Therefore the four-step algorithm is proposed. Computational complexity especially in real case study can be relaxed by executing the following steps and ultimately reduce the computational costs.

3.1 Algorithm

Step 1: Define triangular fuzzy numbers and its respective linguistic variables
The triangular fuzzy numbers is based on a three-value judgment of a linguistic variable. The minimum possible value is denoted as *a*, the most possible value denoted as b and the maximum possible value denoted as c.

Step 2: Delineate Inverse Function
$f_A^L:[a,b]\to[0,w]$ $f_A^R:[c,d]\to[0,w]$. Since $f_A^L:[a,b]\to[0,w]$ is continuous and strictly increasing, the inverse function of f_A^L exists. Similarly, since $f_A^R:[c,d]\to[0,w]$ is continuous and strictly decreasing, the inverse function of f_A^R also exists. The inverse functions of f_A^L and f_A^R can be denoted by g_A^L and g_A^R respectively. Since $f_A^L:[a,b]\to[0,w]$ is continuous and strictly increasing, $g_A^L:[0,w]\to[a,b]$ is continuous and strictly increasing,. Similarly, since $f_A^R:[c,d]\to[0,w]$ is continuous and strictly decreasing, $g_A^R:[0,w]\to[c,d]$ is continuous and strictly decreasing, the inverse function of f_A^R also exists. g_A^L and g_A^R are continuous on [0,w].They are integrable on [0,w] That is, both $\int_0^L g_A^L dx$ *and* $\int_0^R g_A^R dy$ exist.

Step 3: Establish Centroid-Point $(\bar{x},\bar{y})$
The centroid point of a fuzzy number corresponds to an *x* value on the horizontal axis and a *y* value on the vertical axis. The centroid point $(\bar{x},\bar{y})$ for a fuzzy number A:

$$\bar{x}(A)=\frac{\int_a^b (xf_A^L)dx+\int_b^c (xf_A^R)dx}{\int_a^b (f_A^L)dx+\int_b^c (f_A^R)dx} \quad \text{and } \bar{y}(A)=\frac{\int_0^w (yg_A^L)dy+\int_0^w (yg_A^R)dy}{\int_0^w (g_A^L)dy+\int_0^w (g_A^R)dy}$$

where f_A^L and f_A^R are the left and right membership functions of fuzzy number A, respectively.

and g_A^L are g_A^R the inverse functions of f_A^L and f_A^R respectively. The area between the centroid point $(\bar{x},\bar{y})$ and original point (0,0) of the fuzzy number A is then defined as $S(A) = \bar{x}.\bar{y}$

where $\bar{x}$. and $\bar{y}$ are the centroid points of fuzzy number A. To rank fuzzy numbers, we know that the importance of the degree of representative location is higher than average height. Based on this concept for any two fuzzy numbers $\bar{A}$ and $\bar{B}$, we have following situations:

$$\text{if } \bar{x}(A) \succ \bar{x}(B) \Rightarrow \bar{A} \succ \bar{B}$$
$$\text{if } \bar{x}(A) \prec \bar{x}(B) \Rightarrow \bar{A} \prec \bar{B}$$

$$\bar{x}(A) = \bar{x}(B) \Rightarrow \begin{cases} \text{if } \bar{y}(A) \succ \bar{y}(B) \Rightarrow \bar{A} \succ \bar{B} \\ \text{if } \bar{y}(A) \prec \bar{y}(B) \Rightarrow \bar{A} \prec \bar{B} \\ \text{if } \bar{y}(A) = \bar{y}(B) \Rightarrow \bar{A} = \bar{B} \end{cases}$$

Example

We want to compare two fuzzy numbers $\tilde{A} = (5,2,1)$ *and* $\tilde{B} = (7,2,3)$ with the revised Tsao's method.Since $\tilde{x}(A) = 8.5$ and $\tilde{x}(B) = 1.1818$, thus the fuzzy number $\tilde{A}$ is bigger than the fuzzy number $\tilde{B}$. The following figure verifies this result.

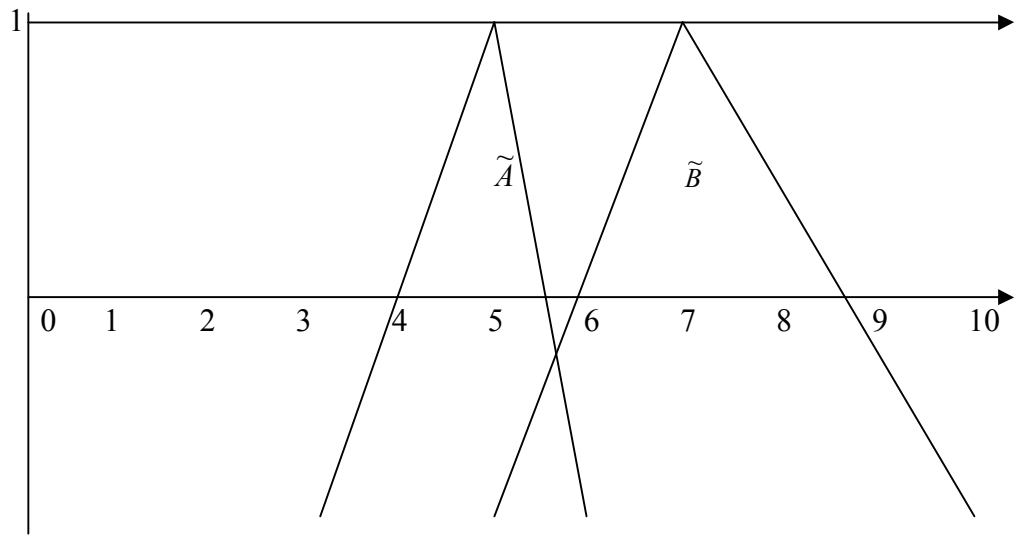

Fig. 2 Comparison Of Two Fuzzy numbers

4 Fuzzy Linear Programming Problem and Breaking Points

Definition 7.

A fuzzy linear programming problem is defined as follows:

Maximize $\tilde{Z} = \tilde{C}\tilde{X}$
subject to $\tilde{A}\tilde{X} \approx \tilde{b}$
$\tilde{X} \geq 0$
where $\tilde{A} \in T(R^{m\times n})$, $b \in T(R^m)$ and $c \in T(R^n)$

Definition 8.
Any $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n) \in T(R)$ where each $x\ i \in T$(R), which satisfies the constraints and non negative restrictions of Fuzzy Linear Programming Problem is said to be fuzzy feasible solution to Fuzzy Linear Programming Problem .

Definition 9.
Consider the system AX = b with $\tilde{X} \geq 0$, where A is an m × n matrix and b is an m vector. Suppose that rank(A) = m. Arranging the column of A as [B, N] that B is an m × m matrix. The vector X = (X_B^T, X_N^T) where $X_B = B^{-1}b$ is called a basic feasible solution (BFS) of system. Here B is called the basic matrix and N is called the non basic matrix. The components of *B* are called the basic variables. If $X_B > 0$, thenX is called a non degenerate basic feasible solution. If $X_B = 0$, then *X* is called a degenerate basic feasible solution.

Definition 10.
A fuzzy basic feasible solution $\tilde{X}_B$ to the Fuzzy linear programming problem is called an optimum fuzzy basic feasible solution if $\tilde{Z}_0 = \tilde{C}_B\tilde{X}_B \geq \tilde{Z}*$, where $\tilde{Z}*$ is the value of objective function for any fuzzy feasible solution.

4.1 Breaking Points

Let us define breaking points of constraints at satisfactory level γ as follows:

That is, $a_{ij} = c - \gamma(c-b)$.By means of the twice intersection of $a_{ij} = c - \gamma(c-b)$ lines, the N = m.n piece values of γ ($0 \leq \gamma \leq 1$) are obtained and then these values change the order of a_{ij}. These values of γ are called the breaking points [10].The order of a_{ij}'s in each subinterval that are formed between the iterative breaking points do not change.

Therefore, the representative point can be selected in this interval. This selected point can be any point of the interval, but the optimal solution of linear programming problem does not change.

However, when the interval changes, the ordering among constraints change. Therefore, the optimal solution of linear programming problem changes also. Because of this, for each interval that is formed the iterative breaking points of γ are searched the optimal solutions one by one. Thus, all optimal solutions are investigated.

Formation of Breaking Points:

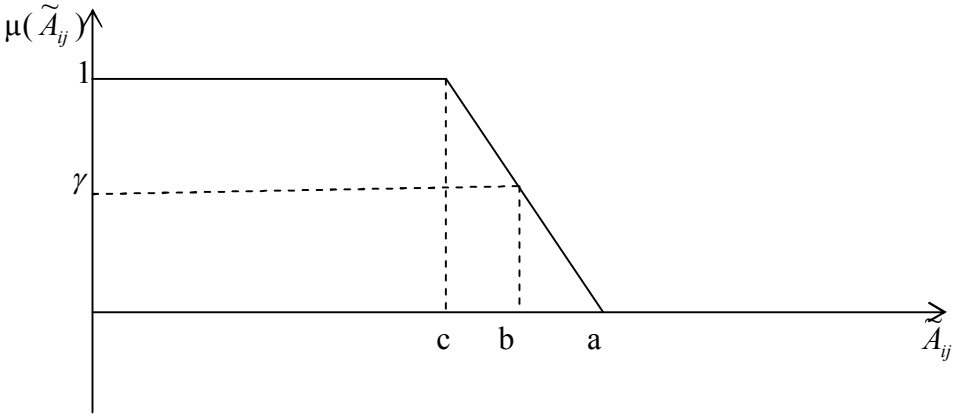

Fig . 3 Breaking points

Let this interval [γ_{p-1}, γ_p] (p = 1,2,…) be subinterval by forming iterative breaking points. By selecting γ_p in [γ_{p-1}, γ_p] , we solve the linear programming problem for crisp coefficients, by using $a_{ij} = c - \gamma(c-b)$.

4.2 Algorithm for new simplex method using breaking points and ranking method

1. Standardizing the given fuzzy linear programming problem by introducing slack or surplus variables.
2. Obtain basic feasible solution to the problem by using $X_B = B^{-1}b$.
3. Using breaking points, rearrange the coefficient of constraints as $a_{ij} = c - \gamma(c-b)$.
4. Find the entering variable using Revised Tsao's method.
5. Find the leaving variable using min $\{\frac{x_{bi}}{y_{ir}}, y_{ir} > 0, i = 1,2,\ldots,m\}$ and hence find the pivotal element which is common to both entering variable column and leaving variable row.
6. Convert the leading element to unity by dividing its row by pivotal element and all elements in its column to zero.
7. If all $z_j - c_j \geq 0$, then the optimal solution is obtained .Otherwise go to step(4) and repeat the procedure till the solution is obtained.

4.3 Numerical Example

Solve

Max $\tilde{z} = (5,2,2)\tilde{x}_1 + (3,1,1)\tilde{x}_2$

s. t. constraints

$$\mathbf{(3,2,3)}\ \tilde{x}_1 + \mathbf{(5,3,2)}\ \tilde{x}_2 \leq \mathbf{(15,5,5)}$$

$$\mathbf{(5,3,4)}\ \tilde{x}_1 + \mathbf{(2,1,1)}\ \tilde{x}_2 \leq \mathbf{(10,5,5)}$$

$$\tilde{x}_1,\ \tilde{x}_2 \geq \mathbf{0}$$

Standardize the linear programming problem by introducing slack variables $\tilde{x}_3, \tilde{x}_4 \geq 0$ The breaking points of γ are intersection points of lines $A_{ij} = A_{ij}$ (γ).Breaking values of γ in

(0,1) interval are 0,1/4,2/3,1. Intervals among these iterative values are [0,1/4],[1/4,2/3],[2/3,1].

Table 1 Initial iteration

$\tilde{C}_B$	$\tilde{X}_B$	V $\tilde{X}_B$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
$\tilde{0}$	$\tilde{x}_3$	20 - 5γ	6 - 3 γ	7 - 2 γ	2 – γ	0
$\tilde{0}$	$\tilde{x}_4$	15 - 5 γ	9 - 4 γ	3 - γ	0	2 - γ
Max $\tilde{z}$	$\tilde{0}$					
	$\tilde{z}_j - \tilde{c}_j$		(-5,2,2)	(-3,1,1)	$\tilde{0}$	$\tilde{0}$

Here $\tilde{y}_{01} = \tilde{z}_1 - \tilde{c}_1$ =(-5,2,2) and $\tilde{y}_{02} = \tilde{z}_2 - \tilde{c}_2$ = (-3,1,1) ,Using revised Tsao method for ordering $\tilde{y}_{01}$ and $\tilde{y}_{02}$,we have $\tilde{y}_{01} > \tilde{y}_{02}$ and hence $\tilde{x}_1$ enters the basis and leaving variable is $\tilde{x}_4$ .Here 9 - 4 γ is a pivotal element

Table 2 I –iteration

$\tilde{C}_B$	$\tilde{X}_B$	V $\tilde{X}_B$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
$\tilde{0}$	$\tilde{x}_3$	$\frac{5\gamma^2-50\gamma+90}{9-4\gamma}$	0	$\frac{5\gamma^2-31\gamma+45}{9-4\gamma}$	2- γ	$\frac{-3\gamma^2+12\gamma-12}{9-4\gamma}$
(5,2,2)	$\tilde{x}_1$	$\frac{15-5\gamma}{9-4\gamma}$	1	$\frac{3-\gamma}{9-4\gamma}$	0	$\frac{2-\gamma}{9-4\gamma}$
Max $\tilde{z}$		$\left(\frac{75-25\gamma}{9-4\gamma},\frac{30-10\gamma}{9-4\gamma},\frac{30-10\gamma}{9-4\gamma}\right)$				
	$\tilde{z}_j - \tilde{c}_j$		$\tilde{0}$	$\left(\frac{7\gamma-12}{9-4\gamma},\frac{-6\gamma+15}{9-4\gamma},\frac{-6\gamma+15}{9-4\gamma}\right)$	$\tilde{0}$	$\left(\frac{10-5\gamma}{9-4\gamma},\frac{4-\gamma}{9-4\gamma},\frac{4-\gamma}{9-4\gamma}\right)$

Proceeding as above we get the final iteration as

Table 3 Final Iteration

$\tilde{C}_B$	$\tilde{X}_B$	V $\tilde{X}_B$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
(3,1,1)	$\tilde{x}_2$	$\frac{5\gamma^2-50\gamma+90}{5\gamma^2-31\gamma+45}$	0	1	$\frac{(2-\gamma)(9-4\gamma)}{5\gamma^2-31\gamma+45}$	$\frac{-3\gamma^2+12\gamma-12}{5\gamma^2-31\gamma+45}$
(5,2,2)	$\tilde{x}_1$	$\frac{-20\gamma^3+165\gamma^2-450\gamma+405}{(9-4\gamma)(5\gamma^2-31\gamma+45}$	1	0	$\frac{(\gamma-2)(3-\gamma)}{5\gamma^2-31\gamma+45}$	$\frac{-8\gamma^3+62\gamma^2-110\gamma+126}{(5\gamma^2-31\gamma+45)(9-4\gamma)}$

$\widetilde{C}_B$	$\widetilde{X}_B$	$\mathrm{V}\,\widetilde{X}_B$	$\widetilde{x}_1$	$\widetilde{x}_2$	$\widetilde{x}_3$	$\widetilde{x}_4$
Max $\widetilde{z}$		$\frac{-160\gamma^3+1260\gamma^2-4680\gamma+4455}{(9-4\gamma)(5\gamma^2-31\gamma+45)}$, $\frac{-60\gamma^3+475\gamma^2-1710\gamma+1620}{(9-4\gamma)(5\gamma^2-31\gamma+45)}$, $\frac{-60\gamma^3+475\gamma^2-1710\gamma+1620}{(9-4\gamma)(5\gamma^2-31\gamma+45)}$				
	$\widetilde{z}_j-\widetilde{c}_j$		$\widetilde{0}$	$\widetilde{0}$	$\frac{17\gamma^2-76\gamma+84}{5\gamma^2-31\gamma+45}$, $\frac{6\gamma^2-27\gamma+50}{5\gamma^2-31\gamma+45}$, $\frac{6\gamma^2-27\gamma+50}{5\gamma^2-31\gamma+45}$	$\frac{-4\gamma^3+85\gamma^2-82\gamma+306}{(9-4\gamma)(5\gamma^2-31\gamma+45)}$, $\frac{20\gamma^3-101\gamma^2+258\gamma-72}{(9-4\gamma)(5\gamma^2-31\gamma+45)}$, $\frac{20\gamma^3-101\gamma^2+258\gamma-72}{(9-4\gamma)(5\gamma^2-31\gamma+45)}$

Table 4 solution

γ \ x	[0,1.4]	[1.4,2.3]	[2.3,1]
$\widetilde{x}_1$	$\frac{5\gamma^2-50\gamma+90}{5\gamma^2-31\gamma+45}$	$\frac{5\gamma^2-50\gamma+90}{5\gamma^2-31\gamma+45}$	$\frac{5\gamma^2-50\gamma+90}{5\gamma^2-31\gamma+45}$
$\widetilde{x}_2$	$\frac{-20\gamma^3+165\gamma^2-450\gamma+405}{(9-4\gamma)(5\gamma^2-31\gamma+45}$	$\frac{-20\gamma^3+165\gamma^2-450\gamma+405}{(9-4\gamma)(5\gamma^2-31\gamma+45}$	$\frac{-20\gamma^3+165\gamma^2-450\gamma+405}{(9-4\gamma)(5\gamma^2-31\gamma+45}$

5 Conclusion

In this paper we have shown that there is an optimal solution in each interval constituting by the iterative breaking points. After the breaking points of the satisfactory level that are changed, the values of decision variables are determined. However, the breaking points of satisfactory levels depend on quantities and the changed optimal solutions are obtained successively.

References

1. Zimmermann. H. J., (1978) .Fuzzy programming and linear programming with Several objective functions. Fuzzy Sets and Systems, 1, 45-55.

2. Delgado, M., Verdegay, J. L. and Vila, M. A., (1989). A general model for fuzzy linear programming. Fuzzy Sets and Systems, 29, 21-29.
3. Fang, S. C. and Hu, C. F., (1999). Linear programming with fuzzy coefficients in constraint. Comput. Math. Appl, 37, 63-76.
4. Mahdavi-Amiri, N., Nasseri, S. H., (2005). Duality in fuzzy variable linear Programming. 4th World Enformatika Conference, WEC'05, June 24- 26, Istanbul, Turkey.
5. Maleki, H. R., (2002). Ranking functions and their applications to fuzzy linear Programming. Far East J. Math. Sci., 4, 283-301.
6. Maleki, H. R., Tata, M., Mashinchi, M., (2000). Linear programming with fuzzy variables. Fuzzy Sets and Systems., 109, 21-33.
7. Rommelfanger, H., Hanuscheck, R., Wolf, J., (1989). Linear programming with fuzzy objective. Fuzzy Sets and Systems, 29, 31- 48.
8. Verdegay, J. L., (1984). A dual approach to solve the fuzzy linear programming Problem. Fuzzy Sets and Systems., 14, 131-141.
9. Nasseri, S. H., Sohrabi, M., (2010). Solving Fuzzy Linear Programming by Using Tsao's Method. Australian Journal of Basic and Applied Sciences, 4(10), 4592-4600.
10. Ringuest, J. L., Rink, D. B., (1987). Interactive solutions for the linear multi objective transportation problem.European Journal Research., 32, 96-106.
11. Bellman, R. E., Zadeh, L. A., (1970). Decision making in a fuzzy environment, Management Sci., 17, 141-164.
12. Guzel, N., (2010). Fuzzy Transportation problem with the fuzzy Amounts and the Fuzzy Costs. World Applied Sciences Journal .8(5), 543-549.
13. Bazaraa, M. S., Jarvis, J. J., Sherali, H. D., (1990). Linear Programming and Network Flows, John Wiley, New York, Second Edition.