A new method for solving fully fuzzy linear Bilevel programming problems

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Abstract In this paper, a new method is proposed to find the fuzzy optimal solution of fully fuzzy linear Bilevel programming (FFBLP) problems by representing all the parameters as triangular fuzzy numbers. In the proposed method, the given FFLBLP problem is decomposed into three crisp linear programming (CLP) problems with bounded variables constraints, the three CLP problems are solved separately and by using its optimal solutions, the fuzzy optimal solution to the given FFLBLP is obtained. The proposed method is easy to understand and to apply for finding the fuzzy optimal solution of FFLBLP occurring in real life situations.

Keywords Fully Fuzzy Linear Programming Problems, Triangular fuzzy Numbers, Bilevel Programming.

1 Introduction

Fuzzy set theory has been applied to many disciplines such as control theory and management sciences, mathematical modeling and industrial applications. The concept of fuzzy mathematical programming on general level was first proposed by Tanaka et al. [1] in the framework of the fuzzy decision of Bellman and Zadeh [2]. The first formulation of fuzzy linear programming (FLP) is proposed by Zimmermann [3]. Many researchers adopted this concept for solving fuzzy linear programming problems [1–14]. However, in all of the above mentioned works, those cases of fuzzy linear programming have been studied in which not all parts of the problem were assumed to be fuzzy, e.g., only the right hand side or the objective function coefficients were fuzzy but the variables were not fuzzy.


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This paper is organized as follows: In Section 2 some basic definitions and arithmetics between two triangular fuzzy numbers are reviewed. In Section 3 formulation of FLBLP problems and a new method is proposed for solving FLBLP problems we solve an illustrative numerical example in Section 4.

2 Preliminaries

In this section, some necessary backgrounds and notions of fuzzy set theory are reviewed.

2.1 Basic definitions

Definition 2.1 [15]. The characteristic function $\mu_A$ of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in $X$. This function can be generalized to a function $\mu_A$ such that the value assigned to the element of the universal set $X$ fall within a specified range i.e. $\mu_A : X \to [0,1]$. The assigned value indicate the membership grade of the element in the set $A$.

The function $\mu_A$ is called the membership function and the set $\tilde{A} = \{ (x, \mu_A(x)) ; x \in X \}$ defined by $\mu_A(x)$ for each $x \in X$ is called a fuzzy set.

Definition 2.2 [15]. A fuzzy number $\tilde{A} = (a, b, c)$ is said to be a triangular fuzzy number if its membership function is given by

$$
\mu_A(x) = \begin{cases} 
\frac{x-a}{b-a} & a \leq x \leq b \\
1 & x = b \\
\frac{x-c}{b-c} & b \leq x \leq c 
\end{cases}
$$

Definition 2.3 [15]. A triangular fuzzy number $(a, b, c)$ is said to be non-negative fuzzy number iff $a \geq 0$.

Definition 2.4 [15]. A fuzzy number $\tilde{A}$ is said to be non-negative fuzzy number if and only if $\mu_A(x) = 0$, for all $x < 0$. 

Definition 2.5 [15]. Two triangular fuzzy numbers $\tilde{A} = (a, b, c)$ and $\tilde{B} = (e, f, g)$ are said to be equal if and only if $a = e, b = f, c = g$.

2.2 Arithmetic operations

In this subsection, arithmetic operations between two triangular fuzzy numbers, defined on universal set of real numbers, are reviewed [4].

Let $\tilde{A} = (a, b, c)$ and $\tilde{B} = (e, f, g)$ be two triangular fuzzy numbers then

(i) $\tilde{A} \oplus \tilde{B} = (a, b, c) \oplus (e, f, g) = (a + e, b + f, c + g)$

(ii) $-\tilde{A} = -(a, b, c) = (-c, -b, -a)$

(iii) Let $\tilde{A} = (a, b, c)$ be any triangular fuzzy number and Let $\tilde{B} = (x, y, z)$ be a non-negative triangular fuzzy number then

$$\tilde{A} \otimes \tilde{B} = \begin{cases} (ax, by, cz) & a \geq 0 \\ (cx, by, az) & a < 0, c \geq 0 \\ (ax, by, cx) & c < 0 \end{cases}$$

3 Fully fuzzy linear bilevel programming problem

Consider a FlBLPP of maximization-type objective functions at each level. Suppose that $DM_i$ denotes the $DM_i$ at the $i$-th level $(i = 1, 2)$ who controls the decision vector

$$\max_{\tilde{x}_1} Z_1 \approx \tilde{C}_{11}^T \tilde{x}_1 \ominus \tilde{C}_{12}^T \tilde{x}_2 \ominus \tilde{a}_1$$

where $\tilde{x}_2$, solve

$$\max Z_2 \approx \tilde{C}_{21}^T \tilde{x}_1 \ominus \tilde{C}_{22}^T \tilde{x}_2 \ominus \tilde{a}_2$$

s.t.

$$\tilde{A} \tilde{x}_1 \oplus \tilde{B} \tilde{x}_2 \{\leq, \approx, \geq\} I\tilde{t}$$

$$\tilde{x}_1, \tilde{x}_2 \geq 0$$

Where $\tilde{a}_1$ and $\tilde{a}_2$ are fuzzy number, $\tilde{A} = (\tilde{a}_y)_{m \times n_1}, \tilde{B} = (\tilde{b}_q)_{m \times n_2}, \tilde{t} = (\tilde{t}_i)_{m \times 1}$ and $\tilde{C}_{11} = (C_{11}), \tilde{C}_{12} = (C_{21}), \tilde{C}_{21} = (C_{12}), \tilde{C}_{22} = (C_{22}), n_1 + n_2 = n, \tilde{a}_y, \tilde{b}_q, \tilde{x}_1, \tilde{x}_2, \tilde{t}_i \in F(R)$

for all $1 \leq i \leq m, 1 \leq j \leq n$.

Let the parameters $\tilde{C}_{11}, \tilde{C}_{12}, \tilde{C}_{21}, \tilde{C}_{22}, \tilde{a}_y, \tilde{b}_q, \tilde{x}_1, \tilde{x}_2$ and $\tilde{t}_i$ be the triangular fuzzy number $(p_{ij}, q_{ij}, r_{ij}), (p_{ij}, q_{ij}, r_{ij}), (p_{ij}, q_{ij}, r_{ij}), (p_{ij}, q_{ij}, r_{ij}), (a_{ij}, a_{ij}, a_{ij}), (b_{ij}, b_{ij}, b_{ij}), (c_{ij}, c_{ij}, c_{ij})$, and $(m_i, n_i, p_i)$ respectively. Then, the problem can be written as follows:
\[
\begin{align*}
\text{Max} & \quad (Z_{1L}, Z_{1M}, Z_{1U}) = \sum_{j=1}^{n_1} (p_{ij}, q_{ij}, r_{ij}) \otimes (x_{ij}, y_{ij}, t_{ij}) \oplus \sum_{j=1}^{n_2} (p_{2j}, q_{2j}, r_{2j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \\
\text{where} & \quad x_{2j}, y_{2j}, t_{2j} \text{ solves} \\
\text{Max} & \quad (Z_{2L}, Z_{2M}, Z_{2U}) = \sum_{j=1}^{n_1} (p_{3j}, q_{3j}, r_{3j}) \otimes (x_{ij}, y_{ij}, t_{ij}) \oplus \sum_{j=1}^{n_2} (p_{4j}, q_{4j}, r_{4j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \\
\text{s.t.} & \quad \sum_{j=1}^{n_1} (a_{ij}^{-1}, a_{ij}^{2}, a_{ij}^{3}) \oplus \sum_{j=1}^{n_2} (b_{ij}^{-1}, b_{ij}^{2}, b_{ij}^{3}) \otimes (x_{ij}, y_{ij}, t_{ij}) \{\leq, =, \geq\} (m, n, p, ) \\
\text{for all} & \quad i = 1, \ldots, m. \\
\text{and all decision variables are non-negative.}
\end{align*}
\]

Now, using the arithmetic operations and partial ordering relations, we decompose the given FLBLPP as follows:

\[
\begin{align*}
\text{Max} & \quad Z_{1L} = \text{lower value of} \quad \sum_{j=1}^{n_1} (p_{ij}, q_{ij}, r_{ij}) \otimes (x_{ij}, y_{ij}, t_{ij}) \oplus \sum_{j=1}^{n_2} (p_{2j}, q_{2j}, r_{2j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \\
\text{Max} & \quad Z_{1M} = \text{middle value of} \quad \sum_{j=1}^{n_1} (p_{ij}, q_{ij}, r_{ij}) \otimes (x_{ij}, y_{ij}, t_{ij}) \oplus \sum_{j=1}^{n_2} (p_{2j}, q_{2j}, r_{2j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \\
\text{Max} & \quad Z_{1U} = \text{upper value of} \quad \sum_{j=1}^{n_1} (p_{ij}, q_{ij}, r_{ij}) \otimes (x_{ij}, y_{ij}, t_{ij}) \oplus \sum_{j=1}^{n_2} (p_{2j}, q_{2j}, r_{2j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \\
\text{where} & \quad x_{2j}, y_{2j}, t_{2j} \text{ solves} \\
\text{Max} & \quad Z_{2L} = \text{lower value of} \quad \sum_{j=1}^{n_1} (p_{3j}, q_{3j}, r_{3j}) \otimes (x_{ij}, y_{ij}, t_{ij}) \oplus \sum_{j=1}^{n_2} (p_{4j}, q_{4j}, r_{4j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \\
\text{Max} & \quad Z_{2M} = \text{middle value of} \quad \sum_{j=1}^{n_1} (p_{3j}, q_{3j}, r_{3j}) \otimes (x_{ij}, y_{ij}, t_{ij}) \oplus \sum_{j=1}^{n_2} (p_{4j}, q_{4j}, r_{4j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \\
\text{Max} & \quad Z_{2U} = \text{upper value of} \quad \sum_{j=1}^{n_1} (p_{3j}, q_{3j}, r_{3j}) \otimes (x_{ij}, y_{ij}, t_{ij}) \oplus \sum_{j=1}^{n_2} (p_{4j}, q_{4j}, r_{4j}) \otimes (x_{2j}, y_{2j}, t_{2j}) \\
\text{s.t.} & \quad \text{lower value of} \quad \sum_{j=1}^{n_1} (a_{ij}^{-1}, a_{ij}^{2}, a_{ij}^{3}) \oplus \sum_{j=1}^{n_2} (b_{ij}^{-1}, b_{ij}^{2}, b_{ij}^{3}) \otimes (x_{ij}, y_{ij}, t_{ij}) \{\leq, =, \geq\} m_i \\
\text{middle value of} \quad \sum_{j=1}^{n_1} (a_{ij}^{-1}, a_{ij}^{2}, a_{ij}^{3}) \oplus \sum_{j=1}^{n_2} (b_{ij}^{-1}, b_{ij}^{2}, b_{ij}^{3}) \otimes (x_{ij}, y_{ij}, t_{ij}) \{\leq, =, \geq\} n_i \\
\text{upper value of} \quad \sum_{j=1}^{n_1} (a_{ij}^{-1}, a_{ij}^{2}, a_{ij}^{3}) \oplus \sum_{j=1}^{n_2} (b_{ij}^{-1}, b_{ij}^{2}, b_{ij}^{3}) \otimes (x_{ij}, y_{ij}, t_{ij}) \{\leq, =, \geq\} p_i \\
\text{for all} & \quad i = 1, \ldots, m. \\
\text{and all decision variables are non-negative.}
\end{align*}
\]

From the above decomposition problem, we construct the following CLP problems namely, middle level bilevel problem (MLBLP), upper level bilevel problem (ULBLP) and lower level bilevel problem (LLBLP) as follows:
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\[ (MLBLP) \]

\[ \max_{z_{1m}} Z_{1m} = \text{middle value of} \left[ \sum_{j=1}^{n} (p_{1j}, q_{1j}, r_{1j}) \odot (x_{1j}, y_{1j}, t_{1j}) \right] \odot \left[ \sum_{j=1}^{n} (p_{2j}, q_{2j}, r_{2j}) \odot (x_{2j}, y_{2j}, t_{2j}) \right] \]

where \( y_{2j} \) solve

\[ \max_{z_{2m}} Z_{2m} = \text{middle value of} \left[ \sum_{j=1}^{n} (p_{3j}, q_{3j}, r_{3j}) \odot (x_{1j}, y_{1j}, t_{1j}) \right] \odot \left[ \sum_{j=1}^{n} (p_{4j}, q_{4j}, r_{4j}) \odot (x_{2j}, y_{2j}, t_{2j}) \right] \]

s.t.

Constraints in the decomposition problem in which at least one decision variable of the MLBLP occurs and all decision variables are non-negative.

\[ (LLBLP) \]

\[ \max_{z_{1l}} Z_{1l} = \text{lower value of} \left[ \sum_{j=1}^{n} (p_{1j}, q_{1j}, r_{1j}) \odot (x_{1j}, y_{1j}, t_{1j}) \right] \odot \left[ \sum_{j=1}^{n} (p_{2j}, q_{2j}, r_{2j}) \odot (x_{2j}, y_{2j}, t_{2j}) \right] \]

where \( x_{2j} \) solve

\[ \max_{z_{2l}} Z_{2l} = \text{lower value of} \left[ \sum_{j=1}^{n} (p_{3j}, q_{3j}, r_{3j}) \odot (x_{1j}, y_{1j}, t_{1j}) \right] \odot \left[ \sum_{j=1}^{n} (p_{4j}, q_{4j}, r_{4j}) \odot (x_{2j}, y_{2j}, t_{2j}) \right] \]

s.t.

\( x_{1j} \leq y_{1j}^{*}, x_{2j} \leq y_{2j}^{*} \)

Constraints in the decomposition constraints in which at least one decision variable of the LLBLP occurs which are not used in MLBLP and ULBLP; all variables in the constraints and objective function in LLBLP must satisfy the bounded constraints; replacing all values of the decision variables which are obtained in the MLBLP and all decision variables are non-negative. Where \( y_{1j}^{*}, y_{2j}^{*} \) is the optimal value of MLBLP.

\[ (ULBLP) \]

\[ \max_{z_{ul}} Z_{ul} = \text{lower value of} \left[ \sum_{j=1}^{n} (p_{1j}, q_{1j}, r_{1j}) \odot (x_{1j}, y_{1j}, t_{1j}) \right] \odot \left[ \sum_{j=1}^{n} (p_{2j}, q_{2j}, r_{2j}) \odot (x_{2j}, y_{2j}, t_{2j}) \right] \]

where \( t_{2j} \) solve

\[ \max_{z_{ul}} Z_{ul} = \text{upper value of} \left[ \sum_{j=1}^{n} (p_{3j}, q_{3j}, r_{3j}) \odot (x_{1j}, y_{1j}, t_{1j}) \right] \odot \left[ \sum_{j=1}^{n} (p_{4j}, q_{4j}, r_{4j}) \odot (x_{2j}, y_{2j}, t_{2j}) \right] \]

s.t.

\( t_{1j} \geq y_{1j}^{*}, t_{2j} \geq y_{2j}^{*} \)

Constraints in the decomposition problem in which at least one decision variable of the ULBLP occurs and are not used in MLBLP; all variables in the constraints and objective function in ULBLP must satisfy the bounded constraints; replacing all values of the decision variables which are obtained in MLBLP.
and LLBLP and all decision variables are non-negative. Where \( y_{1j}^*, y_{2j}^* \) is the optimal value of MLBLP.

4 Numerical example

Example: Consider the following fully fuzzy linear bilevel programming problem:

Max \( Z_i \approx (3, 5, 7) \odot \bar{x}_i \oplus (2, 4, 8) \odot \bar{y}_i \)

where \( \bar{x}_i \) solve

Max \( Z_2 \approx (3, 5, 10) \odot \bar{x}_1 \oplus (1, 7, 8) \odot \bar{x}_2 \)

s.t.

\[(4, 5, 9) \odot \bar{x}_1 \oplus (2, 7, 8) \odot \bar{x}_2 \leq (4, 10, 20)\]

\[(0, 3, 7) \odot \bar{x}_1 \oplus (1, 2, 10) \odot \bar{x}_2 \leq (2, 5, 18)\]

\( \bar{x}_1, \bar{x}_2 \geq 0 \)

Now, the decomposition problem of the given FLPP is given below:

Max \( Z_{IL} = 3x_1 + 2x_2 \)

Max \( Z_{IM} = 5y_1 + 4y_2 \)

Max \( Z_{IU} = 7t_1 + 8t_2 \)

where \( x_1, y_2 \) and \( t_2 \) solves

Max \( Z_{IL} = 3x_1 + x_2 \)

Max \( Z_{IM} = 5y_1 + 7y_2 \)

Max \( Z_{IU} = 10t_1 + 8t_2 \)

s.t.

\[4x_1 + 2x_2 \leq 4, 5y_1 + 7y_2 \leq 10, 9t_1 + 8t_2 \leq 20\]

\[x_2 \leq 2, 3y_1 + 2y_2 \leq 5, 7t_1 + 10t_2 \leq 18\]

\[x_i, y_i, t_i \geq 0, \quad i = 1, 2.\]

Now, the Middle Level problem is given below:

\[(MLBLP)\]

Max \( Z_{1M} = 5y_1 + 4y_2 \)

where \( y_2 \) solve

Max \( Z_{2M} = 5y_1 + 7y_2 \)

s.t.

\[5y_1 + 7y_2 \leq 10, 3y_1 + 2y_2 \leq 5\]

\[y_1, y_2 \geq 0\]
Now, solving the problem (MLBLP) using simplex method, we obtain the optimal solution \( y_1 = 1.36, y_2 = 0.45 \) and \( Z_{\text{ML}} = 8.6 \).

Now, the lower Level problem is given below:

\[
\text{(LBLLP)} \quad \begin{align*}
\text{Max} & \quad Z_{\text{LL}} = 3x_1 + 2x_2 \\
\text{where} & \quad x_2 \text{ solve} \\
\text{Max} & \quad Z_{\text{2L}} = 3x_1 + x_2 \\
\text{s.t.} & \quad 4x_1 + 2x_2 \leq 4, x_2 \leq 2 \\
& \quad x_1 \leq 1.36, x_2 \leq 0.45 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Now, solving the problem LLP with \( y_1 = 1.36, y_2 = 0.45 \) by simplex method, we obtain the optimal solution \( x_1 = 0.39, x_2 = 0.45 \) and \( Z_{\text{LL}} = 2.07 \).

Now, the Upper Level problem is given below:

\[
\text{(ULBLP)} \quad \begin{align*}
\text{Max} & \quad Z_{\text{UL}} = 7t_1 + 8t_2 \\
\text{where} & \quad t_2 \text{ solves} \\
\text{Max} & \quad Z_{\text{2U}} = 10t_1 + 8t_2 \\
\text{s.t.} & \quad 9t_1 + 8t_2 \leq 20, 7t_1 + 10t_2 \leq 18 \\
& \quad t_1 \geq 1.36, t_2 \geq 0.45 \\
& \quad t_1, t_2 \geq 0
\end{align*}
\]

Now, solving the problem LLP with \( y_1 = 1.36, y_2 = 0.45 \) by simplex method, we obtain the optimal solution \( t_1 = 1.65, t_2 = 0.65 \) and \( Z_{\text{UL}} = 16.75 \).

Therefore, an optimal fuzzy solution to the given fully fuzzy linear Bilevel programming problem is
\[
\bar{x}_1 = (x_1, y_1, t_1) = (0.39, 1.36, 1.65), \bar{x}_2 = (x_2, y_2, t_2) = (0.45, 0.45, 0.45), \bar{Z}_1 = (Z_{\text{UL}}, Z_{\text{1M}}), \\
Z_{\text{{1M, Z_{\text{UL}}}}} = (2.07, 8.6, 16.75) \text{ and } \bar{Z}_2 = (Z_{\text{2L}}, Z_{\text{2M}}, Z_{\text{2U}}) = (1.62, 9.95, 21.7).
\]

5 Conclusions

In this paper, fuzzy linear Bilevel programming problems with fuzzy variables and fuzzy constraints are discussed. The significant of this paper is solving fuzzy linear Bilevel programming problem when the variables triangular fuzzy numbers without using ranking method. Thus the method is very useful in the real world problems where the product is uncertain.
References