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An Optimal Method Based on Rationalized Haar Wavelet for Approximate Answer of Stochastic Ito-Volterra Integral Equations

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Abstract This article proposes an optimal method for approximate answer of stochastic Ito-Voltrra integral equations, via rationalized Haar functions and their stochastic operational matrix of integration. Stochastic Ito-voltreea integral equation is reduced to a system of linear equations. This scheme is applied for some examples. The results show the efficiency and accuracy of the method.

Keywords: Rationalized Haar Functions, Stochastic Operational Matrix, Product Matrix, Stochastic Ito-Volterra Integral Equations, Ito-Integral, Brownian Motion Process.

1 Introduction

Stochastic Ito-volterra integral equations arise in many applications such as mathematical finance, biology, medical, social sciences, etc. there is an increasing demand for studying the behavior of a number of sophisticated dynamical systems in physical, medical and social sciences, as well as in engineering, finance and population growth [1]. These systems are often dependent on a noise source, on a Gaussian white noise, for example, governed by certain probability laws. So that modelling such phenomena naturally requires the use of various stochastic differential equations [2-7] or, in more complicated cases, stochastic Ito-volterra integral equations and stochastic integro – differential equations [8-17]. Because in many problems such equations of course cannot be solved explicitly, it is important, to find their approximate solutions by using some numerical methods [2-5, 13-15].

Many orthogonal functions or polynomials, such as Block Pulse functions, Hybrid functions, Haar wavelet, Legendre wavelet, Coifman wavelet, Shannon wavelet, Daubechies wavelet, and Bernestein polynomials, were used to derive solutions of different integral equations, [18-22]. Here we use the rationalized haar wavelet and stochastic integration operational matrix for derive solution of stochastic Ito-volterra integral equation.

So, consider the following linear stochastic Ito-volterra integral equation,

$$X(t) = f(t) + \int_0^t b(t,s)X(s)ds + \int_0^t \sigma(t,s)X(s)dB(s) \qquad t \in [0,T),$$

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Where *X(t), f(t), b(t, s)* and $\sigma(t,s)$, for $t,s \in [0,T)$, are the stochastic processes defined on the same probability space (Ω, F, P), and *X(t)* is unknown function. *B(t)* is a Brownian motion and $\int_0^t \sigma(t,s)X(s)dB(s)$ is the Ito-integral.

This paper is organized as follows,

In Section 2, we describe the basic properties of the rationalized Haar functions and functions approximation by rationalized Haar functions and integration operational matrix. In section3, we introduce concept of the stochastic integration operational matrix based on rationalized Haar functions. In Section4, we solve stochastic Ito-Volterra integral equations by using stochastic integration operational matrix. In section5, we examine the efficiency and accuracy of this method by giving some numerical examples. Finally, section 6 gives some brief conclusion.

2 Rationalized Haar Functions (RHFs)

The goal of this section is to recall notations and definition of the rationalized Haar functions and recall some known results and formulas that are important for this paper. These have discussed thoroughly in [21, 22].

2.1 Definition

The rationalized Haar functions (RHFs) is defined as:

$$RH(r,t) = \begin{cases} 1 & \dfrac{j-1}{2^i} \le t < \dfrac{j-\frac{1}{2}}{2^i}, \\ -1 & \dfrac{j-\frac{1}{2}}{2^i} \le t < \dfrac{j}{2^i}, \\ 0 & otherwise. \end{cases} \tag{1}$$

where

$r = 3^i + j - 1 \quad , \quad i = 0,1,2,3,... \quad , \quad j = 1,2,3,...,2^i.$

RH(0,t) is defined for i = j = 0 and is given by

$$RH(0,t) = 1 \quad , \quad 0 \le t < 1 \, , \tag{2}$$

With orthogonality property,

$$\int_0^1 RH(r,t)\, RH(\upsilon,t)dt = \begin{cases} 2^{-i} & for \quad r = \upsilon, \\ 0 & for \quad r \ne \upsilon. \end{cases} \tag{3}$$

Where

$\upsilon = 2^n + m - 1 \quad , \quad n = 0,1,2,3,... \quad , \quad m = 1,2,3,...,2^n.$

2.2 Functions approximation

A functions $f(t)$ defined over the interval $t \in [0,1]$ may be expanded in RHFs as,

$$f(t) = \sum_{r=o}^{\infty} f_r RH(r,t), \tag{4}$$

where $f_r, r = 0,1,2,...,$ are given by

$$f_r = 2^i \int_0^1 f(t) RH(r,t) dt, \tag{5}$$

with $r = 2^i + j - 1$ for $i = 0,1,2,3,...,$ $j = 1,2,3,...,2^i$, and $r = 0$ for $i = j = 0$.

If we let $i = 0,1,2,...,\alpha,$ then the infinite series in equation (4) is truncated up to its first k terms as,

$$f(t) \cong \sum_{r=0}^{k-1} f_r RH(r,t) = F^T \Phi(t) = \Phi^T(t) F, \tag{6}$$

where $k = 2^{\alpha+1}, \alpha = 0,1,2,....$

The vectors of F and $\Phi(t)$ are defined as,

$$F = (f_0, f_1, ..., f_{k-1})^T, \tag{7}$$

$$\Phi(t) = \left(\phi_0(t), \phi_1(t), ..., \phi_{k-1}(t)\right)^T, \quad \phi_r(t) = RH(r,t), \quad r = 0,1,2,...,k-1. \tag{8}$$

Let $k(t,s) \in L^2\left([0,1) \times [0,1)\right)$. It can be similarly expanded with respect to RHFs such as,

$$k(t,s) \cong \sum_{r=0}^{k-1} \sum_{\upsilon=0}^{k-1} k_{r\upsilon} \phi_r(t) \phi_\upsilon(s) = \Phi^T(t) K \Phi(s), \tag{9}$$

where $K = (k_{r\upsilon})_{k \times k}$, and $k_{r\upsilon}$ for $r = 0,1,3,...,k-1, \upsilon = 0,1,2,...,k-1$, is given by

$$k_{r\upsilon} = 2^{i+n} \int_0^1 \int_0^1 k(t,s) \phi_r(t) \phi_\upsilon(s) dt ds, \qquad i,n = 0,1,3,...,\alpha.$$

The first eight RHFs can be written in matrix form as,

$$\hat{\Phi}_{8\times 8} = \begin{pmatrix} \phi_0(t) \\ \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \\ \phi_4(t) \\ \phi_5(t) \\ \phi_6(t) \\ \phi_7(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{10}$$

In equation (10), the row denotes the order of the Haar function. The matrix $\hat{\Phi}_{k\times k}$ can be expressed as

$$\hat{\Phi}_{k\times k} = \left(\Phi(\frac{1}{2k}), \Phi(\frac{3}{2k}), \Phi(\frac{5}{2k}), ... \Phi(\frac{2k-1}{2k}) \right), \tag{11}$$

and using equation (6) we get

$$\left(f(\frac{1}{2k}), f(\frac{3}{2k}), f(\frac{5}{2k}), ..., f(\frac{2k-1}{2k})\right) = F^T \hat{\Phi}_{k\times k}. \tag{12}$$

Form equations (9) and (12) we have.

$$K = \left(\hat{\Phi}_{k\times k}^{-1}\right)^T \hat{K} \hat{\Phi}_{k\times k}^{-1}, \tag{13}$$

where

$$\hat{K} = \left(\hat{k}_{lp}\right)_{k\times k}, \quad \hat{k}_{lp} = k\left(\frac{2l-1}{2k}, \frac{2p-1}{2k}\right), \quad l, p = 1,2,3,...,k,$$

and so,

$$\hat{\Phi}_{k\times k}^{-1} = \left(\frac{1}{k}\right)\hat{\Phi}_{k\times k}^{T} .diag(1,1,2,2,\underbrace{2^2,...,2^2}_{2^2},\underbrace{2^3,...,2^3}_{2^3},...,\underbrace{\frac{k}{2},...,\frac{k}{2}}_{\frac{k}{2}}). \tag{14}$$

2.3 The product operational matrix

The rationalized Haar product matrix is defined by [22],

$$\Psi_{k\times k}(t) = \Phi(t)\Phi^T(t). \tag{15}$$

Furthermore by (1) and (2) we get,

$$\phi_0(t)\phi_q(t) = \phi_q(t), \qquad q = 0,1,2,...,k-1,$$

And for $p < q$, we can write

$$\phi_p(t)\phi_q(t) = \begin{cases} \phi_q(t) & if \quad \phi_q(t)\, occurs\, during\, the\, first\;\; positive\, half\;\, wave\;\, of\;\; \phi_p(t), \\ -\phi_q(t) & if \quad \phi_q(t)\, occurs\;\; during\;\; the\; \sec ond\;\; negative\;\; half\;\; wave\;\; of\;\; \phi_p(t), \\ 0 & \qquad otherwise. \end{cases} \tag{16}$$

Also, the square of any RHFs is a block pulse functions, with magnitude of 1 during both the positive and negative half waves of RHFs. Thus, we get,

$$\Psi_{8\times 8}(t) = \begin{pmatrix} \phi_0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_7 \\ \phi_1 & \phi_0 & \phi_2 & -\phi_3 & \phi_4 & \phi_5 & -\phi_6 & -\phi_7 \\ \phi_2 & \phi_2 & \frac{\phi_0+\phi_1}{2} & 0 & \phi_4 & -\phi_5 & 0 & 0 \\ \phi_3 & -\phi_3 & 0 & \frac{\phi_0-\phi_1}{2} & 0 & 0 & \phi_6 & -\phi_7 \\ \phi_4 & \phi_4 & \phi_4 & 0 & \frac{\phi_0+\phi_1+2\phi_2}{4} & 0 & 0 & 0 \\ \phi_5 & \phi_5 & -\phi_5 & 0 & 0 & \frac{\phi_0+\phi_1-2\phi_2}{4} & 0 & 0 \\ \phi_6 & -\phi_6 & 0 & \phi_6 & 0 & 0 & \frac{\phi_0-\phi_1+2\phi_3}{4} & 0 \\ \phi_7 & -\phi_7 & 0 & -\phi_7 & 0 & 0 & 0 & \frac{\phi_0-\phi_1-2\phi_3}{4} \end{pmatrix} \quad (17$$

In general we have,

$$\Psi_{k\times k}(t)=\begin{pmatrix}\Psi_{(\frac{k}{2})\times(\frac{k}{2})}(t) & H_{(\frac{k}{2})\times(\frac{k}{2})}(t)\\ H^T_{(\frac{k}{2})\times(\frac{k}{2})}(t) & D_{(\frac{k}{2})\times(\frac{k}{2})}(t)\end{pmatrix}, \tag{18}$$

where

$\Psi_{1\times 1}(t)=\phi_0(t),$

$$H_{(\frac{k}{2})\times(\frac{k}{2})}(t)=\hat{\Phi}_{(\frac{k}{2})\times(\frac{k}{2})}.diag\left(\phi_{\frac{k}{2}}(t),\phi_{\frac{k}{2}+1}(t),...,\phi_{k-1}(t)\right),$$

$$D_{(\frac{k}{2})\times(\frac{k}{2})}(t)=diag\left[\hat{\Phi}^{-1}_{(\frac{k}{2})\times(\frac{k}{2})}\cdot\left(\phi_0(t),\phi_1(t),...,\phi_{\frac{k}{2}-1}(t)\right)^T\right]^T.$$

Furthermore, by multiplying the matrix $\Psi_{k\times k}(t)$ by the vector F in equation (7), we obtain,

$$\Psi_{k\times k}(t)F=\hat{F}_{k\times k}\Phi(t), \tag{19}$$

where

$$\hat{F}_{k\times k}=\begin{pmatrix}\hat{F}_{(\frac{k}{2})\times(\frac{k}{2})} & G_{(\frac{k}{2})\times(\frac{k}{2})}\\ \hat{G}_{(\frac{k}{2})\times(\frac{k}{2})} & \hat{D}_{(\frac{k}{2})\times(\frac{k}{2})}\end{pmatrix}, \tag{20}$$

with

$\hat{F}_{1\times 1}(t)=f_0,$

$$G_{(\frac{k}{2})\times(\frac{k}{2})}=\Phi_{(\frac{k}{2})\times(\frac{k}{2})}.diag\,(f_{\frac{k}{2}},f_{\frac{k}{2}+1},...,f_{k-1}),$$

$$\hat{G}_{(\frac{k}{2})\times(\frac{k}{2})}=diag\,(f_{\frac{k}{2}},f_{\frac{k}{2}+1},...,f_{k-1}).\hat{\Phi}^{-1}_{(\frac{k}{2})\times(\frac{k}{2})},$$

$$\hat{D}_{(\frac{k}{2})\times(\frac{k}{2})}=diag\left[(f_0,f_1,...,f_{\frac{k}{2}-1}).\hat{\Phi}_{(\frac{k}{2})\times(\frac{k}{2})}\right].$$

2.4 Integration operational matrix

Consider the following approximation

$$\int_0^t\Phi(s)ds\cong P\Phi(t), \tag{21}$$

with operational matrix of integration,

$$P_{k\times k}=\frac{1}{2k}\begin{pmatrix}2kP_{(\frac{k}{2})\times(\frac{k}{2})} & -\hat{\Phi}_{(\frac{k}{2})\times(\frac{k}{2})}\\ \hat{\Phi}^{-1}_{(\frac{k}{2})\times(\frac{k}{2})} & 0\end{pmatrix}, \tag{22}$$

where $\hat{\Phi}_{1\times1}=1$ and $P_{1\times1}=\frac{1}{2}$.

So, we can write

$$\int_0^t f(s)ds \cong \int_0^t F^T\Phi(s)ds \cong F^T P\Phi(t). \tag{23}$$

Also the integration of cross product of two RH function vector is,

$$\int_0^1 \Phi(t)\Phi^T(t)dt = D, \tag{24}$$

where D is diagonal matrix given by

$$D = diag(1,1,\frac{1}{2},\frac{1}{2},\underbrace{\frac{1}{2^2},...,\frac{1}{2^2}}_{2^2},\underbrace{\frac{1}{2^3},...,\frac{1}{2^3}}_{2^3},...,\underbrace{\frac{1}{2^\alpha},...,\frac{1}{2^\alpha}}_{2^\alpha}).$$

3 Stochastic integration operational matrix

Here we would like to compute the Ito integral for each $\phi_r(t), r=0,1,2,...,k-1$. To illustrate the calculation procedures, first, let $\alpha=0$ or $k=2$. using equations (1) and (2) we get,

$$\int_0^t \phi_0(s)dB(s) = B(t) \cong \begin{cases} B(\frac{1}{4}) & 0 \le t < \frac{1}{2}, \\ B(\frac{3}{4}) & \frac{1}{2} \le t < 1, \end{cases} \tag{25}$$

$$\int_0^t \phi_1(s)dB(s) = \begin{cases} B(t) & 0 \le t < \frac{1}{2}, \\ 2B(\frac{1}{2}) - B(t) & \frac{1}{2} \le t < 1, \end{cases} \cong \begin{cases} B(\frac{1}{4}) & 0 \le t < \frac{1}{2}, \\ 2B(\frac{1}{2}) - B(\frac{3}{4}) & \frac{1}{2} \le t < 1. \end{cases} \tag{26}$$

We can rewrite (25) and (26), in terms of the RHFs $\phi_0(t)$ and $\phi_1(t)$, as follows

$$\int_0^t \Phi(s)dB(s) \cong S\Phi(t), \tag{27}$$

Where 2×2 stochastic operational matrix of integration, is given by

$$S_{2\times2} = \frac{1}{2}\begin{pmatrix} B(\frac{1}{4})+B(\frac{3}{4}) & B(\frac{1}{4})-B(\frac{3}{4}) \\ B(\frac{1}{4})-B(\frac{3}{4})+2B(\frac{1}{2}) & B(\frac{1}{4})+B(\frac{3}{4})-2B(\frac{1}{2}) \end{pmatrix}. \tag{28}$$

For convenience, consider

$$\alpha_{pq} = \sum_{i=\frac{(q-1)k}{p}+1}^{\frac{qk}{p}} B(\frac{2i-1}{2k}) \ , \ p=1,2,2^2,2^3,...,k \ , \quad q=1,2,3,4,...,p, \tag{29}$$

and

$$\beta_{pq} = \alpha_{pq} - \alpha_{p+q+1}, \quad p=1,2,2^2,2^3,...,k, \qquad q=1,2,3,4,...,p-1. \tag{30}$$

By using equations (29) and (30), $S_{2\times2}$, is written by

$$S_{2\times 2}=\frac{1}{2}\begin{pmatrix}\alpha_{11} & \beta_{21}\\ \beta_{21}+2B(\frac{1}{2}) & \alpha_{11}-2B(\frac{1}{2})\end{pmatrix}. \tag{31}$$

Now, we choose $\alpha=1$ or $k=4$. using equations (1) and (2) we get the following consecutive relations,

$$\int_0^t \phi_0(s)dB(s)=B(t)\cong\begin{cases}B(\frac{1}{8}) & 0\le t<\frac{1}{4},\\ B(\frac{3}{8}) & \frac{1}{4}\le t<\frac{1}{2},\\ B(\frac{5}{8}) & \frac{1}{2}\le t<\frac{3}{4},\\ B(\frac{7}{8}) & \frac{3}{4}\le t<1,\end{cases} \tag{32}$$

$$\int_0^t \phi_1(s)dB(s)=\begin{cases}B(t) & 0\le t<\frac{1}{2},\\ 2B(\frac{1}{2})-B(t) & \frac{1}{2}\le t<1,\end{cases}\cong\begin{cases}B(\frac{1}{8}) & 0\le t<\frac{1}{4},\\ B(\frac{3}{8}) & \frac{1}{4}\le t<\frac{1}{2},\\ 2B(\frac{1}{2})-B(\frac{5}{8}) & \frac{1}{2}\le t<\frac{3}{4}\\ 2B(\frac{1}{2})-B(\frac{7}{8}) & \frac{3}{4}\le t<1,\end{cases} \tag{33}$$

$$\int_0^t \phi_2(s)dB(s)=\begin{cases}B(t) & 0\le t<\frac{1}{4}\\ 2B(\frac{1}{4})-B(t) & \frac{1}{4}\le t<\frac{1}{2},\\ 2B(\frac{1}{4})-B(\frac{1}{2}) & \frac{1}{2}\le t<1,\end{cases}\cong\begin{cases}B(\frac{1}{8}) & 0\le t<\frac{1}{4},\\ 2B(\frac{1}{4})-B(\frac{3}{8}) & \frac{1}{4}\le t<\frac{1}{2},\\ 2B(\frac{1}{4})-B(\frac{1}{2}) & \frac{1}{2}\le t\,\frac{3}{4},\\ 2B(\frac{1}{4})-B(\frac{1}{2}) & \frac{3}{4}\le t<1.\end{cases} \tag{34}$$

$$\int_0^t \phi_3(s)dB(s)=\begin{cases}0 & 0\le t<\frac{1}{2}\\ B(t)-B(\frac{1}{2}) & \frac{1}{2}\le t<\frac{3}{4}\\ 2B(\frac{3}{4})-B(\frac{1}{2})-B(t) & \frac{3}{4}\le t<1,\end{cases}\cong\begin{cases}0 & 0\le t<\frac{1}{2}\\ B(\frac{5}{8})-B(\frac{1}{2}) & \frac{1}{2}\le t<\frac{3}{4}\\ 2B(\frac{3}{4})-B(\frac{1}{2})-B(\frac{7}{8}) & \frac{3}{4}\le t<1.\end{cases} \tag{35}$$

We can rewrite (32), (33), (34 and (35), in terms of the RHFs $\phi_0(t)$, $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$, as follows

$$\int_0^t \Phi(s)dB(s)\cong S\,\Phi(t), \tag{36}$$

where 4×4 stochastic operational matrix of integration, is given by

$$S_{4\times 4}=\frac{1}{4}\begin{pmatrix} \alpha_{11} & \beta_{21} & 2\beta_{41} & 2\beta_{43} \\ \beta_{21}+4B(\frac{1}{2}) & \alpha_{11}-4B(\frac{1}{2}) & 2\beta_{41} & -2\beta_{43} \\ \beta_{41}-2B(\frac{1}{2})+6B(\frac{1}{4}) & \beta_{41}+2B(\frac{1}{2})-2B(\frac{1}{4}) & 2\alpha_{21}-4B(\frac{1}{4}) & 0 \\ \beta_{43}-2B(\frac{1}{2})+2B(\frac{3}{4}) & -\beta_{43}+2B(\frac{1}{2})-2B(\frac{3}{4}) & 0 & 2\alpha_{22}-4B(\frac{3}{4}) \end{pmatrix} \tag{37}$$

In general we have,

$$S_{k\times k}=\frac{1}{k}\begin{pmatrix} C_{(\frac{k}{2})\times(\frac{k}{2})} & U_{(\frac{k}{2})\times(\frac{k}{2})} \\ V_{(\frac{k}{2})\times(\frac{k}{2})} & \bar{D}_{(\frac{k}{2})\times(\frac{k}{2})} \end{pmatrix}, \tag{38}$$

where

$$U_{(\frac{k}{2})\times(\frac{k}{2})}=\frac{k}{2}\hat{\Phi}_{(\frac{k}{2})\times(\frac{k}{2})}.diag(\beta_{k1},\beta_{k3},\beta_{k5},\ldots,\beta_{k,k-1}),$$

$$\bar{D}_{(\frac{k}{2})\times(\frac{k}{2})}=\frac{k}{2}diag\left(\alpha_{\frac{k}{2},1}-2B(\frac{1}{k}),\alpha_{\frac{k}{2},2}-2B(\frac{3}{k}),\alpha_{\frac{k}{2},3}-2B(\frac{5}{k}),\ldots,\alpha_{\frac{k}{2},\frac{k}{2}}-2B(\frac{k-1}{k})\right),$$

And

$$C_{1\times 1}=\alpha_{11},\qquad C_{(\frac{k}{2})\times(\frac{k}{2})}=\frac{k}{2}\hat{S}_{(\frac{k}{2})\times(\frac{k}{2})},$$

where $\hat{S}_{(\frac{k}{2})\times(\frac{k}{2})}$ is the same matrix $S_{(\frac{k}{2})\times(\frac{k}{2})}$, with this difference that all coefficients $B(\frac{2i-1}{2p})$, for $p=1,2,2^2,2^3,\ldots,\frac{k}{2}, i=1,2,3,\ldots,p$ are doubled. Namely for $k=4,8$ we have,

$$C_{2\times 2}=2\hat{S}_{2\times 2}=\begin{pmatrix} \alpha_{11} & \beta_{21} \\ \beta_{21}+4B(\frac{1}{2}) & \alpha_{11}-4B(\frac{1}{2}) \end{pmatrix},$$

$$C_{4\times 4}=4\hat{S}_{4\times 4}=\begin{pmatrix} \alpha_{11} & \beta_{21} & 2\beta_{41} & 2\beta_{43} \\ \beta_{21}+8B(\frac{1}{2}) & \alpha_{11}-8B(\frac{1}{2}) & 2\beta_{41} & -2\beta_{43} \\ \beta_{41}-4B(\frac{1}{2})+12B(\frac{1}{4}) & \beta_{41}+4B(\frac{1}{2})-4B(\frac{1}{4}) & 2\alpha_{21}-8B(\frac{1}{4}) & 0 \\ \beta_{43}-4B(\frac{1}{2})+4B(\frac{3}{4}) & -\beta_{43}+4B(\frac{1}{2})-4B(\frac{3}{4}) & 0 & 2\alpha_{22}-8B(\frac{3}{4}) \end{pmatrix},$$

and so,

$$V_{1\times 1}=\beta_{21}+2B(\frac{1}{2}),$$

$$V_{2\times 2}=\begin{pmatrix} \beta_{41}-2B(\frac{1}{2})+6B(\frac{1}{4}) & \beta_{41}+2B(\frac{1}{2})-2B(\frac{1}{4}) \\ \beta_{43}-2B(\frac{1}{2})+2B(\frac{3}{4}) & -\beta_{43}+2B(\frac{1}{2})-2B(\frac{3}{4}) \end{pmatrix},$$

$$V_{3\times 3} = \begin{pmatrix} \beta_{81} - 6B(\frac{1}{4}) + 14B(\frac{1}{8}) & \beta_{81} + 2B(\frac{1}{4}) - 2B(\frac{1}{8}) & 2\beta_{81} + 4B(\frac{1}{4}) - 4B(\frac{1}{8}) & 0 \\ \beta_{83} - 6B(\frac{1}{4}) + 10B(\frac{3}{8}) - 4B(\frac{1}{2}) & \beta_{83} + 2B(\frac{1}{4}) - 6B(\frac{3}{8}) + 4B(\frac{1}{2}) & -2\beta_{83} + 4B(\frac{1}{4}) - 4B(\frac{3}{8}) & 0 \\ \beta_{85} - 4B(\frac{1}{2}) + 6B(\frac{5}{8}) - 2B(\frac{3}{4}) & -\beta_{85} + 4B(\frac{1}{2}) - 6B(\frac{5}{8}) + 2B(\frac{3}{4}) & 0 & 2\beta_{85} - 4B(\frac{5}{8}) + 4B(\frac{3}{4}) \\ \beta_{87} - 2B(\frac{3}{4}) + 2B(\frac{7}{8}) & -\beta_{87} + 2B(\frac{3}{4}) - 2B(\frac{7}{8}) & 0 & -2\beta_{87} + 4B(\frac{3}{4}) - 4B(\frac{7}{8}) \end{pmatrix}.$$

So, the Ito integral of every function *f(t)* can be approximated as follows

$$\int_0^t f(s)dB(s) = \int_o^t F^T \Phi(s)dB(s) \cong F^T S\Phi(t). \qquad (39)$$

4 Solving stochastic Ito – Volterra integral equations by using stochastic operational matrix

Consider the following linear stochastic Ito – volterra integral equation,

$$X(t) = f(t) + \int_o^t b(t,s)X(s)\,ds + \int_0^t \sigma(t,s)X(s)dB(s), \quad t \in [0,1), \qquad (40)$$

where *X(t), f(t), b(t, s)* and $\sigma(t,s)$, for $t,s \in [0,1)$, are the stochastic processes defined on the same probability space (Ω, F, P), and *X(t)* is unknown function. Also *B(t)* is a Brownian motion and $\int_0^t \sigma(t,s)X(s)dB(s)$ is the Ito integral.

By using (6), (9) we have below consecutive approximations

$$X(t) \cong X^T \Phi(t) = \Phi^T(t)X,$$

$$f(t) \cong F^T \Phi(t) = \Phi^T(t)F,$$

$$b(t,s) \cong \Phi^T(t)B\Phi(s) = \Phi^T(s)B^T\Phi(t),$$

$$\sigma(t,s) \cong \Phi^T(t)\Sigma\Phi(s) = \Phi^T(s)\Sigma^T\Phi(t).$$

In the above approximations, *X* and *F* are the RHFs coefficient stochastic vector, and *B* and Σ are the RHFs coefficient stochastic matrix.

With substituting above approximation in equation (40), we get

$$X^T\Phi(t) \cong F^T\Phi(t) + X^T\left(\int_0^t \Phi(s)\Phi^T(s)ds\right)B^T\Phi(t) + X^T\left(\int_0^t \Phi(s)\Phi^T(s)dB(s)\right)\Sigma^T\Phi(t). \qquad (41)$$

Let b_j be the *j*th column of the constant matrix B, and σ_j be the *j*th column of the constant matrix Σ, and p_i be the *i*th row of the integration operational matrix *P*, s_i be the *i*th row of the stochastic integration operational matrix *S*.

To illustrate the calculation procedures, we choose $\alpha = 1$ or $k = 4$. Using equations (15), (18), (19), (20) and (21) we get,

$$\left(\int_0^t \Phi(s)\Phi^T(s)ds\right)B^T\Phi(t) = \left(\int_0^t \Psi_{4\times 4}(s)ds\right)B^T\Phi(t)$$

$$= \begin{pmatrix} p_1\Phi(t) & p_2\Phi(t) & p_3\Phi(t) & p_4\Phi(t) \\ p_2\Phi(t) & p_1\Phi(t) & p_3\Phi(t) & -p_4\Phi(t) \\ p_3\Phi(t) & p_3\Phi(t) & \frac{p_1+p_2}{2}\Phi(t) & 0 \\ p_4\Phi(t) & -p\Phi(t) & 0 & \frac{p_1-p_2}{2}\Phi(t) \end{pmatrix} \begin{pmatrix} b_1^T\Phi(t) \\ b_2^T\Phi(t) \\ b_3^T\Phi(t) \\ b_4^T\Phi(t) \end{pmatrix}$$

$$=\begin{pmatrix} p_1\Phi(t)\Phi^T(t)b_1+p_2\Phi(t)\Phi^T(t)b_2+p_3\Phi(t)\Phi^T(t)b_3+p_4\Phi(t)\Phi^T(t)b_4 \\ p_2\Phi(t)\Phi^T(t)b_1+p_1\Phi(t)\Phi^T(t)b_2+p_3\Phi(t)\Phi^T(t)b_3-p_4\Phi(t)\Phi^T(t)b_4 \\ p_3\Phi(t)\Phi^T(t)b_1+p_3\Phi(t)\Phi^T(t)b_2+\frac{p_1+p_2}{2}\Phi(t)\Phi^T(t)b_3 \\ p_4\Phi(t)\Phi^T(t)b_1-p_4\Phi(t)\Phi^T(t)b_2+\frac{p_1-p_2}{2}\Phi(t)\Phi^T(t)b_4 \end{pmatrix}$$

$$=\begin{pmatrix} p_1\hat{B}_1+p_2\hat{B}_2+p_3\hat{B}_3+p_4\hat{B}_4 \\ p_2\hat{B}_1+p_1\hat{B}_2+p_3\hat{B}_3-p_4\hat{B}_4 \\ p_3\hat{B}_1+p_3\hat{B}_2+\frac{p_1+p_2}{2}\hat{B}_3 \\ p_4\hat{B}_1-p_4\hat{B}+\frac{p_1-p_2}{2}\hat{B}_4 \end{pmatrix}\Phi(t)$$

$$=\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_1 & p_3 & -p_4 \\ p_3 & p_3 & \frac{p_1+p_2}{2} & 0 \\ p_4 & -p_4 & 0 & \frac{p_1-p_2}{2} \end{pmatrix}\begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \\ \hat{B}_4 \end{pmatrix}\Phi(t)$$

$$=\overline{P}_{4\times4}\overline{B}_{4\times1}\Phi(t)=E_{4\times4}\Phi(t).$$

In general we have,

$$\left(\int_0^t\Phi(s)\Phi^T(s)ds\right)B^T\Phi(t)=E_{k\times k}\Phi(t), \tag{42}$$

where

$$E_{k\times k}=\overline{P}_{k\times k}\overline{B}_{k\times1},$$

with

$$\overline{P}_{k\times k}=\begin{pmatrix} \overline{P}_{(\frac{k}{2})\times(\frac{k}{2})} & \overline{H}_{(\frac{k}{2})\times(\frac{k}{2})} \\ \overline{H}^T_{(\frac{k}{2})\times(\frac{k}{2})} & \Omega_{(\frac{k}{2})\times(\frac{k}{2})} \end{pmatrix}, \tag{43}$$

where

$$\overline{P}_{1\times1}=p_1,$$

$$H_{(\frac{k}{2})\times(\frac{k}{2})}=\hat{\Phi}_{(\frac{k}{2})\times(\frac{k}{2})}.diag\left(p_{\frac{k}{2}+1},p_{\frac{k}{2}+2},\dots,p_k\right),$$

$$\Omega_{(\frac{k}{2})\times(\frac{k}{2})}=diag\left[\hat{\Phi}^{-1}_{(\frac{k}{2})\times(\frac{k}{2})}.(p_1,p_2,\dots,p_{\frac{k}{2}})^T\right]^T.$$

and

$$B_{k\times1}=\begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \vdots \\ \hat{B}_k \end{pmatrix}.$$

Similarly,

$$\left(\int_0^t \Phi(s)\Phi^T(s)dB(s)\right)\Sigma^T \Phi(t) = \hat{E}_{k\times k}\Phi(t), \tag{44}$$

where

$$\hat{E}_{k\times k} = \widetilde{S}_{k\times k}\widetilde{\Sigma}_{k\times 1},$$

with

$$\widetilde{S}_{k\times k} = \begin{pmatrix} \widetilde{S}_{(\frac{k}{2})\times(\frac{k}{2})} & M_{(\frac{k}{2})\times(\frac{k}{2})} \\ M^T_{(\frac{k}{2})\times(\frac{k}{2})} & \Lambda_{(\frac{k}{2})\times(\frac{k}{2})} \end{pmatrix}, \tag{45}$$

where

$$S_{1\times 1} = s_1,$$

$$M_{(\frac{k}{2})\times(\frac{k}{2})} = \hat{\Phi}_{(\frac{k}{2})\times(\frac{k}{2})}.diag\left(s_{\frac{k}{2}+1}, s_{\frac{k}{2}+2}, \dots, s_k\right),$$

$$\Lambda_{(\frac{k}{2})\times(\frac{k}{2})} = diag\left[\hat{\Phi}^{-1}_{(\frac{k}{2})\times(\frac{k}{2})}.\left(s_1, s_2, \dots, s_{\frac{k}{2}}\right)^T\right]^T$$

and

$$\Sigma_{k\times 1} = \begin{pmatrix} \hat{\Sigma}_1 \\ \hat{\Sigma}_2 \\ \vdots \\ \hat{\Sigma}_k \end{pmatrix}.$$

With substituting relations (42) and (44) in (41), we get

$$X^T\Phi(t) \cong F^T\Phi(t) + X^T E\Phi(t) + X^T \hat{E}\Phi(t).$$

Then,

$$X^T(I - E - \hat{E}) \cong F^T. \tag{46}$$

So, by setting $N = (I - E - \hat{E})^T$ and replacing $\cong$ by =, we will have,

$$NX = F. \tag{47}$$

Which is a linear system of equations that gives the approximate RH functions coefficient of the unknown stochastic processes *X(t)*, so

$$X(t) \cong X^T\Phi(t).$$

5 Numerical examples

In this section, we present a selection of examples to illustrate the efficiency of the method proposed in Section 4.

Example 1. *Consider the following linear stochastic Ito – Volterra integral equation,*

$$S(t) = \frac{1}{75} + \int_0^t \ln(1+s)S(s)ds + \int_0^t sS(s)dB(s) \qquad s,t \in [0,1), \tag{48}$$

with the exact solution $S(t)=\frac{1}{75}e^{(1+t)\ln(1+t)-t-\frac{t^3}{6}+\int_0^t sdB(s)}$, for $0\le t<1$, where *S(t)* is and unknown stochastic processes defined on the probability space (Ω,F,P), and *B(t)* is a Brownian motion process. The approximate solutions for $k=2,4,8,16$ and exact solution are shown in Table 1.

Table 1 The approximate solutions for $k=2,4,8,16$ and exact solution

t	$k=2$	$k=4$	$k=8$	$k=16$	Exact solution
0	0.012446	0.012506	0.013538	0.013300	0.013333
0.1	0.012446	0.012506	0.013538	0.013570	0.014032
0.2	0.012446	0.012506	0.012177	0.014007	0.014605
0.3	0.012446	0.020903	0.015059	0.012808	0.012908
0.4	0.012446	0.020903	0.018561	0.016015	0.013614
0.5	0.009464	0.009475	0.015079	0.014059	0.014736
0.6	0.009464	0.009475	0.015079	0.015269	0.011277
0.7	0.009464	0.009475	0.011106	0.017104	0.012905
0.8	0.009464	0.015404	0.008001	0.013479	0.012024
0.9	0.009464	0.015404	0.010052	0.008253	0.009568
1	0.009464	0.015404	0.010052	0.007738	0.008951
Error	0.0077007	0.0084467	0.0084499	0.0109982	

Example 2. *Consider the following linear stochastic Ito – volterra integral equation,*

$$X(t)=\frac{1}{120}+\int_0^t\cos(s)X(s)ds+\int_0^t\sin(s)X(s)dB(s)\quad s,t\in[0,1), \tag{49}$$

with the exact solution $X(t)=\frac{1}{120}e^{-\frac{t}{4}+\sin(t)+\frac{\sin(2t)}{8}+\int_0^t\sin(s)dB(s)}$, for $0\le t<1$, where *X(t)* is an unknown stochastic processes defined on the probability space (Ω,F,P), and *B(t)* is a Brownian motion process. The approximate solutions for $k=2,4,8,16$ and exact solution are shown in Table 2.

Table 2 The approximate solutions for $k=2,4,8,16$ and exact solution

t	$k=2$	$k=4$	$k=8$	$k=16$	Exact solution
0	0.008788	0.009303	0.009309	0.008954	0.008333
0.1	0.008788	0.009303	0.009309	0.010751	0.009213
0.2	0.008788	0.009303	0.011334	0.012490	0.010558
0.3	0.008788	0.017452	0.011897	0.014532	0.008043
0.4	0.008788	0.017452	0.012129	0.019704	0.009639
0.5	0.010013	0.013618	0.011407	0.013463	0.011868
0.6	0.010013	0.013618	0.011407	0.020933	0.011711
0.7	0.010013	0.013618	0.011816	0.020932	0.010444
0.8	0.010013	0.014921	0.014287	0.019602	0.013001
0.9	0.010013	0.014921	0.008403	0.006376	0.010339
1	0.010013	0.014921	0.008403	0.005247	0.016419
Error	0.0078348	0.0090492	0.0086805	0.0075787	

6 Conclusion

Because for some SDE's that can be written as Volterra integral equations, it is impossible to find the exact solution of Eq. (40), it would be convenient to determine its numerical solution based on stochastic numerical analysis. Using Rationalized Haar functions as basis functions to solve the linear stochastic Ito – Volterra integral equations is very simple and effective in comparison with other methods. Here, the applicability and accuracy of this method discussed in two examples. The results of numerical solution. So, if we encounter with the SDE's similar to Volterra integral equations as we can't solve them analytically, we can use this method.

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