International Journal of Applied Operational Research
Vol. 7, No. 1, pp. 23-31, Winter 2017

Journal homepage: ijorlu.liau.ac.ir

A Suggested Approach for Stochastic Interval-Valued Linear Fractional Programming problem

S. H. Nasseri*, S. Bavandi

Received: 19 May 2016 ; **Accepted:** 20 October 2016;

Abstract In this paper, we considered a Stochastic Interval-Valued Linear Fractional Programming problem(SIVLFP). In this problem, the coefficients and scalars in the objective function are fractional-interval, and technological coefficients and the quantities on the right side of the constraints were random variables with the specific distribution. Here we changed a Stochastic Interval-Valued Fractional Programming problem to an optimization problem with an interval-valued objective function, so that its boundaries are fractional functions. A numerical example was presented to demonstrate the effectiveness of the proposed method.

Keywords: Linear Fractional Programming, Interval-Valued Function, Interval-Valued Linear Fractional Programming, Chance-Constrained Programming.

1 Introduction

During the modeling of real-world practical issues, it was observed that some parameters of the problem were not definitely clear. In particular, for an optimization problem, it is possible that the parameters be imprecise. For instance, the right side quantities in a linear programming problem are imprecise or the coefficients of the objective function are fuzzy [1].

There are several approaches for uncertainty model in optimization problems, such as stochastic and fuzzy optimizations. Here, we considered an optimization problem with an interval-valued objective function, so that the existing coefficients in the constraints are the random variable. Tigan, Minasian, Stancu [2,3] analyzed these kinds of problems. Hsien-Chung wu [4,5] obtained and proved the Karush-Kuhn-Tucker (K.K.T) conditions for each of the optimization problems with interval-valued objective function.

So far, Fractional programming has attracted many researchers. The main reason of its attraction is due to this fact that if we consider the optimization ratio between physical and economical quantities, then the programming models will be more consistent with the real issues. Generally, the fractional optimization is the optimization of one or more ratios [6]. Naturally, these models are created in decision making situations, where the simultaneous optimization of several ratios is required, such as production planning, financial planning,

* **Corresponding Author.** (✉)
E-mail: nasseri@umz.ac.ir **(S. H. Nasseri)**

S. H. Nasseri
Associated Professor, Department of Mathematics, University of Mazandaran, Babolsar, Iran.

S. Bavandi
Master Student, Department of Mathematics, University of Mazandaran, Babolsar, Iran.

healthcare planning, etc. So, fractional planning as an approach to improved certain programming [7], has two advantages: first, it is possible to use different objectives as the uniform objective function consistent with the preservation of its features and without any change in size. Second, it is possible to estimate the system efficiency as the ratio of two magnitudes by designing the objective function, for example, the ratio of costs to returns, the ratio of costs to time, etc. The fractional programming focused mainly on the engineering, economics and environmental management [7].

Several approaches were proposed for the fractional programming problems, such as variable transformation method [8] and the updated objective function [9]. There are some new methods in this field, for further readings see [10-12].

Here, we introduced an interval-valued linear fractional programming problem, and then we transformation it to an optimization problem with an interval-valued target function. After explaining some definitions required for interval account and some introductions about the probability in section 2, we introduced the Stochastic Interval-Valued Fractional Programming problem and also we stated the certain equation forms for the constraints in section 3. In section 4, we solved a numerical example to show the effectiveness of the model and finally we presented some results in section 5.

2 Preliminaries

In this section, we recall some basic definitions and properties of interval arithmetic and Axiomatic Probability.

2.1 Interval Arithmetic

Definition 2.1. An interval is defined by an ordered pair of brackets as

$A = \left[a^L, a^U\right] = \left\{a : a^L \le a \le a^U, a \in \mathbb{R}\right\}$,

where a^L and a^U are the left and right limits of A, respectively. The set of all closed and bounded interval in R will be shown by I.

Denote the center and radius of A respectively as

$$A^C = \frac{1}{2}\left(a^U + a^L\right), \qquad A^R = \frac{1}{2}\left(a^U - a^L\right)$$

Definition 2.2. Let $*$ denote one of the arithmetic operations $+, -, \times$ or $\div$ and let $A = \left[a^L, a^U\right]$ and $B = \left[b^L, b^U\right]$ so that $A, B \in I$, then the generalization of ordinary arithmetic to closed intervals is known as interval arithmetic, and is defined by:

$A * B = \left\{a * b : a \in A, b \in B\right\}$,

where we assume $0 \notin B$ in the case of division.

Let $k \in \mathbb{R}$ be a constant. from Definitions 2.1 and 2.2, we can see that

$A + B = \left[a^L + b^L, a^U + b^U\right]$,

$A - B = \left[a^L - b^U, a^U - b^L\right]$,

$$kA = k\left[a^L, a^U\right] = \begin{cases} \left[ka^L, ka^U\right] & if \ \ k \geq 0, \\ \left[ka^U, ka^L\right] & if \ \ k < 0, \end{cases}$$

$$A.B = \left[min\left\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\right\}, max\left\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\right\}\right],$$

$$A \div B = \left[a^L, a^U\right]\left[\frac{1}{b^U}, \frac{1}{b^L}\right], \qquad 0 \notin B.$$

Definition 2.3. Let $A = \left[a^L, a^U\right]$ and $B = \left[b^L, b^U\right]$ be two closed, bounded, real intervals in I, then we say that $A \leq B$, if and only if $a^L \leq b^L$ and $a^U \leq b^U$
and we say $A < B$ if and only if

$$\begin{cases} a^L < b^L \\ a^U \leq b^U \end{cases} \quad or \quad \begin{cases} a^L \leq b^L \\ a^U < b^U \end{cases} \quad or \quad \begin{cases} a^L < b^L \\ a^U < b^U \end{cases}$$

2.2 Axiomatic Probability

The primary reference for sections 2.2 and 2.3 are [8] and [11].

Probability theory is derived from a small set of axioms and a minimal set of essential assumptions. The first concept in probability theory is the sample space, which is an abstract concept containing primitive probability events.

Definition 2.4. The sample space is a set Ω that contains all possible outcomes.

Definition 2.5. An event ω is a subset of the sample space Ω . An event may be any subsets of the sample space (including the entire sample space), and the set of all events is known as the event space.

Definition 2.6. The set of all events in the sample space Ω is called the event space and is denoted $\mathcal{F}$.

Assembling a sample space, event space and a probability measure into a set produces what is known as a probability space.

Definition 2.7. A probability space is denoted using the tuple $(\Omega, \mathcal{F}, P)$ where Ω is the sample space, $\mathcal{F}$ is the event space and P is the probability set function which has domain $\omega \in \mathcal{F}$.

2.3 Random Variables

This section covers univariate random variables.

Definition 2.8. Let $(\Omega, \mathcal{F}, P)$ be a probability space. If $X : \Omega \to \mathbb{R}$ is a real-valued function have as its domain elements of Ω , then X is called a random variable.

Definition 2.9. A random variable is called discrete if its range consists of a countable (possibly infinite) number of elements.

Discrete random variables are characterized by a Probability Mass Function (PMF) which gives the probability of observing a particular value of the random variable.

Definition 2.10. The probability mass function for a discrete random variable X is defined as $f(x)=P(x)$, for all $x \in R(X)$ and $f(x)=0$, for all $x \notin R(X)$, where $R(X)$ is the range of X (i.e. the values for which X is defined).

Definition 2.11. A random variable is called continuous if its range is uncountably infinite and there exists a non-negative-valued function $f(x)$ defined or all $x \in (-\infty, \infty)$ such that for any event $B \subset R(X), P(X)=\int_{x \in B} f(x)dx$ and $f(x)=0$, for all $x \in R(X)$, where $R(X)$ is the range of X (i.e. the values for which X is defined).

The PMF of a discrete random variable is replaced with the probability density function (pdf) for continuous random variables.

Definition 2.12. For a continuous random variable, the function f is called the Probability Density Function (PDF). A function $f:\mathbb{R}\rightarrow\mathbb{R}$ is a member of the class of continuous density functions if and only if $f(x)\geq 0$, for all $x \in(-\infty,\infty)$ and $\int_{-\infty}^{\infty} f(x)=1$.

Definition 2.13. The Cumulative Distribution Function (CDF) for a random variable X is defined as $F(c)=P(x \leq c)$, for all $c \in(-\infty,\infty)$.

The cumulative distribution function is used for both discrete and continuous random variables. When X is a discrete random variable, the CDF is

$$F(x)=\sum_{s \leq x} f(s),$$

for $c \in (-\infty,\infty)$ and when X is a continuous random variable, the CDF is

$$F(x)=\int_{-\infty}^{x} f(s)ds,$$

for $x \in (-\infty,\infty)$.

2.4 Interval-Valued Function

The reference for this section is [13].

Definition 2.14. A function $f:\mathbb{R}^n \rightarrow I$ is called an interval-valued function (because $f(x)$ for each $x \in \mathbb{R}^n$ is a closed interval in $\mathbb{R}$). Similar to interval notation, we denote the interval valued function f with $f(x)=\left[f^L(x), f^U(x)\right]$ where for every $x \in \mathbb{R}^n$ $f^L(x), f^U(x)$ are real valued functions and $f^L(x) \leq f^U(x)$.

Proposition 2.1. Let f be an interval valued function defined on $\mathbb{R}^n$. Then f is continuous at $c \in \mathbb{R}^n$ if an only if f^L and f^U are continuous at c.

Definition 2.15. Let X be an open set in $\mathbb{R}$. An interval valued function $f:X\rightarrow I$ with $f(x)=\left[f^L(x), f^U(x)\right]$ is called weak differentiable at x_0, if the real valued functions $f^L(x)$ and $f^U(x)$ are differentiable (usual differentiability) at x_0.

Definition 2.16. We define a linear fractional function

$$F(x) = \frac{cx + \alpha}{dx + \beta}$$

where $x = (x_1, x_2, \ldots, x_n)^t \in \mathbb{R}^n$, $c = (c_1, c_2, \ldots, c_n)^t \in \mathbb{R}^n$, $d = (d_1, d_2, \ldots, d_n)^t \in \mathbb{R}^n$ and α, β are real scalars.

3 Stochastic Interval-Valued Linear Fractional Programming(SIVLFP)

Stochastic Interval-Valued Linear Fractional Programming can be formulated as follows

$$\begin{aligned} &Min \ \ z = \frac{\sum_{j=1}^{n} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta} \\ &s.t. \\ &\qquad \sum_{j=1}^{n} a_{ij}^s x_j \le b_i^s, \\ &\qquad x_j \ge 0. \end{aligned} \tag{1}$$

where $c_j, d_j \in I, j = 1, 2, \ldots, n, a_{ij}^s, b_i^s, j = 1, \ldots, n, i = 1, \ldots, m,$ are independent random variables with known distribution functions. We denote c_j^L and d_j^L the lower bounds of the intervals c_j and d_j respectively i.e. $c^L = (c_1^L, c_2^L, ..., c_n^L)$ and also $d^L = (d_1^L, d_2^L, \ldots, d_n^L)$ where c_j^L and c_j^U are real scalars for $j = 1, \ldots, n$ and $x \in \mathbb{R}^n$, similarly we can define c_j^U and d_j^U. Also $\alpha = [\alpha^L, \alpha^U]$, $\beta = [\beta^L, \beta^U]$. From the fact that $a_{ij}^s, j = 1, \ldots, n, i = 1, \ldots, m$ and $b_i^s, i = 1, \ldots, m$ are independent random variables with the normal distribution , thus by incorporating predetermined tolerance measures β_i, that $0 \le \beta_i \le 1, i = 1, \ldots, m$, and by utilizing the chance-constrained approach [13], the set of stochastic constraints of problem (1) can be transformed to their deterministic equivalents as follows

$$\Pr\left(\sum_{j=1}^{n} a_{ij}^s x_j \le b_i^s \right) \ge \beta_i, \ i = 1, ..., m,$$

Let $E(.)$ and $Var(.)$ be the mean and the variance of random variables a_{ij}^s and b_i^s respectively, Therefore we have

$$\sum_{j=1}^{n} E(a_{ij}^s) x_j - \Phi^{-1}(1 - \beta_i) \sqrt{Var(b_i^s) + \sum_{j=1}^{n} Var(a_{ij}^s) x_j^2} \le E(b_i^s), \ \ i = 1, ..., m. \tag{2}$$

where $\Phi^{-1}(.)$ is the inverse distribution function of the standard normal distribution.

So we can rewrite (1) as follows

$$
\begin{aligned}
& Min\ f(x)=\frac{z(x)}{w(x)} \\
& s.t. \\
& \qquad \sum_{j=1}^{n} E\left(a_{ij}^{s}\right)x_j-\Phi^{-1}\left(1-\beta_i\right)\sqrt{Var\left(b_i^{s}\right)+\sum_{j=1}^{n}Var\left(a_{ij}^{s}\right)x_j^2}\le E\left(b_i^{s}\right),\quad i=1,\dots,m, \\
& \qquad x_j\ge 0.
\end{aligned}
\tag{3}
$$

where $z(x), w(x)$ are interval-valued linear functions as

$$z(x)=\left[z^L(x),z^U(x)\right]=\left[c^Lx+\alpha^L,c^Ux+\alpha^U\right]$$

$$w(x)=\left[w^L(x),w^U(x)\right]=\left[d^Lx+\beta^L,d^Ux+\beta^U\right]$$

So we have

$$
\begin{aligned}
& Min\ f(x)=\frac{\left[\sum_{j=1}^{n}c_j^Lx_j+\alpha^L,\sum_{j=1}^{n}c_j^Ux_j+\alpha^U\right]}{\left[\sum_{j=1}^{n}d_j^Lx_j+\beta^L,\sum_{j=1}^{n}d_j^Ux_j+\beta^U\right]} \\
& s.t. \\
& \qquad \sum_{j=1}^{n} E\left(a_{ij}^{s}\right)x_j-\Phi^{-1}\left(1-\beta_i\right)\sqrt{Var\left(b_i^{s}\right)+\sum_{j=1}^{n}Var\left(a_{ij}^{s}\right)x_j^2}\le E\left(b_i^{s}\right),\quad i=1,\dots,m, \\
& \qquad x_j\ge 0.
\end{aligned}
\tag{4}
$$

Also, we can consider another kind of problem (4) as follows

$$
\begin{aligned}
& Min\ f(x)=\left[f^L(x),f^U(x)\right] \\
& s.t. \\
& \qquad \sum_{j=1}^{n} E\left(a_{ij}^{s}\right)x_j-\Phi^{-1}\left(1-\beta_i\right)\sqrt{Var\left(b_i^{s}\right)+\sum_{j=1}^{n}Var\left(a_{ij}^{s}\right)x_j^2}\le E\left(b_i^{s}\right),\quad i=1,\dots,m, \\
& \qquad x_j\ge 0.
\end{aligned}
\tag{5}
$$

Theorem 3.1. Any Stochastic Interval-Valued Linear Fractional Programming problem in the form (4) under some assumptions can be converted to an Interval-Valued Linear Fractional Programming problem in the form (5).

Proof: See [14]

Below definition and theorem took from [14] that will be useful in our discussion**.**

Definition 3.1. Let x^* be a feasible solution to the problem (5). We say that x^* is a nondominated solution of problem (5), if there exist no feasible solution x such that $f(x)<f(x^*)$. In this case, we say that $f\left(x^*\right)$ is the nondominated objective value of f .

Consider the following optimization problem corresponding to problem (5)

$$
\begin{aligned}
& Min \;\; g(x) = f^{L}(x) + f^{U}(x) \\
& s.t. \\
& \qquad \sum_{j=1}^{n} E\left(a_{ij}^{s}\right)x_j - \Phi^{-1}\left(1-\beta_i\right)\sqrt{Var\left(b_i^{s}\right) + \sum_{j=1}^{n} Var\left(a_{ij}^{s}\right)x_j^{2}} \le E\left(b_i^{s}\right), \;\; i = 1,\dots,m, \\
& \qquad x_j \ge 0.
\end{aligned}
\tag{6}
$$

For solve problem (5), we use the theorem.

Theorem3.2. If x^* is an optimal solution of problem (6), then x^* is a nondominated solution of problem (5).

Proof: See [15].

4 Numerical Example

In this section, we solve a numerical example using the proposed method. Consider the following optimization problem with random variable coefficients, where the coefficients of the left-hand sides and the right-hand are independent random variables with the normal distribution:

$$
\begin{aligned}
& Min\, f(x) = \frac{[1,2]x_1 + [3,7]x_2 + \left[\frac{3}{2},\frac{5}{2}\right]x_3 + \left[\frac{7}{2},4\right]}{\left[\frac{1}{2},1\right]x_1 + \left[\frac{3}{4},1\right]x_2 + \left[\frac{7}{8},2\right]x_3 + \left[\frac{1}{2},1\right]} \\
& s.t. \\
& \qquad a_{11}^{s}x_1 + a_{12}^{s}x_2 + a_{13}^{s}x_3 \le b_1^{s}, \\
& \qquad a_{21}^{s}x_1 + a_{22}^{s}x_2 + a_{23}^{s}x_3 \le b_2^{s}, \\
& \qquad a_{31}^{s}x_1 + a_{32}^{s}x_2 + a_{33}^{s}x_3 \le b_1^{s}, \\
& \qquad x_1, x_2, x_3 \ge 0.
\end{aligned}
\tag{7}
$$

For the above normal random interval variables, assume that

$b_1^{s} \sim N\left(6, 3^2\right)$,

$b_2^{s} \sim N\left(8, 1^2\right)$,

$b_3^{s} \sim N\left(13, 0.5^2\right)$.

and

Table 1 The expectations and variances of the technical coefficients

	a_{11}^{s}	a_{12}^{s}	a_{13}^{s}	a_{21}^{s}	a_{22}^{s}	a_{23}^{s}	a_{31}^{s}	a_{32}^{s}	a_{33}^{s}
$E(.)$	1	1	-1	-2	3	1	1	1	1
$Var(.)$	0.5	1	0.5	0.5	0.5	1	1	0.25	0.25

Assume that the Decision Maker (DM), tolerance measures as $\beta_i = 0.8$, $i = 1,2,3$.

Problem (7) by using Theorem 3.1 and expression (2) be transformed into the following equivalent problems

$$
\begin{aligned}
&Min \quad f(x) = \left[\frac{x_1 + 3x_2 + \frac{3}{2}x_3 + \frac{7}{2}}{x_1 + x_2 + 2x_3 + 1}, \frac{2x_1 + 7x_2 + \frac{5}{2}x_3 + 4}{\frac{1}{2}x_1 + \frac{3}{4}x_2 + \frac{7}{8}x_3 + \frac{1}{2}} \right] \\
&s.t. \\
&\quad x_1 + x_2 - x_3 - \Phi^{-1}(0.2)\sqrt{9 + 0.5x_1^2 + x_2^2 + 0.5x_3^2} \le 6, \\
&\quad -2x_1 + 3x_2 + x_3 - \Phi^{-1}(0.2)\sqrt{1 + 0.5x_1^2 + 0.5x_2^2 + x_3^2} \le 8, \\
&\quad x_1 + x_2 + x_3 - \Phi^{-1}(0.2)\sqrt{0.25 + x_1^2 + 0.25x_2^2 + 0.5x_3^2} \le 13, \\
&\quad x_1, x_2, x_3 \ge 0.
\end{aligned} \tag{8}
$$

with regard to the of problem (5), we have

$$
\begin{aligned}
&Min \quad g(x) = \frac{x_1 + 3x_2 + \frac{3}{2}x_3 + \frac{7}{2}}{x_1 + x_2 + 2x_3 + 1} + \frac{2x_1 + 7x_2 + \frac{5}{2}x_3 + 4}{\frac{1}{2}x_1 + \frac{3}{4}x_2 + \frac{7}{8}x_3 + \frac{1}{2}} \\
&s.t. \\
&\quad x_1 + x_2 - x_3 - \Phi^{-1}(0.2)\sqrt{9 + 0.5x_1^2 + x_2^2 + 0.5x_3^2} \le 6, \\
&\quad -2x_1 + 3x_2 + x_3 - \Phi^{-1}(0.2)\sqrt{1 + 0.5x_1^2 + 0.5x_2^2 + x_3^2} \le 8, \\
&\quad x_1 + x_2 + x_3 - \Phi^{-1}(0.2)\sqrt{0.25 + x_1^2 + 0.25x_2^2 + 0.5x_3^2} \le 13, \\
&\quad x_1, x_2, x_3 \ge 0.
\end{aligned} \tag{9}
$$

As a result, a nondominated solution for (7) is $x^* = (0.8088756, 0, 5.140197)$ with $g(x^*) = 4.412911$, which is the optimal solution of (9).

5 Conclusion

In this paper, at first, we introduced two kinds of linear fractional programming problems with interval-valued objective functions and constraints with random coefficients. Then, after imposing some changes, we obtained a nondominated solution for the main linear fractional programming problems with interval-valued objective function. The model proposed in this study was an uncertainty mode where the coefficients of objective function were as interval imprecise values and the coefficients in the constraints and the right side values were a random variable. works to investigate an approach to solving the appropriate nonlinear fractional programming problems for solving the quadratic fractional programmin is in progress. The problem can be studied for cases where the coefficients of the objective function and the coefficients in constraints are either interval or random-interval. It can be also considered in its random-fuzzy mode.

Reference

1. Ohta, H., Yamaguchi, T., (1996). Linear Fractional Goal Programming in Consideration of Fuzzy Solution. European Journal of Operational Research, 92, 157-165.
2. Stancu-Minasian, M., (1984). Stochastic Programming with Multiple Objective Functions. D. Reidel Publishing Company, Dordrecht.
3. Stancu-Minasian, M., Tigan, S. T., (1990). Multiobjective Mathematical Programming with Inexact Data. Kluwer Academic Publishers, Boston, 395-418.
4. Wu, H. C., (2007). The Karush-Kuhn-Tucker Optimality Conditions in an Optimization Problem with Interval-Valued Objective Function. European Journal of Operational Research, 176, 46-59.
5. Wu, H. C., (2008). On Interval-Valued Nonlinear Programming Problems. Journal of Mathematical Analysis and Applications, 299-316.
6. Stancu-Minasian, M., (1997). Fractional Programming. Kluwer Academic Publishers, Dordrecht.
7. Liu, F., Wen, Z., Xu, Y., (2015). A fuzzy fractional chance-constrained programming model for air quality management under uncertainty, Engineering Optimization.
8. Charnes, A., Cooper, W. W., (1962). Programs with Linear Fractional Functions. Naval Research Logistics Quarterly, 3-4, 181-196.
9. Bitran, G. R., Novaes, A. G., (1973). Linear Programming with a Fractional Function. Operations Research, 21, 22-29.
10. Jiao, H., Guo, Y., Shen, P., (2006). Global Optimization of Generalized Linear Fractional Programming with Nonlinear Constraints. Applied Mathematics and Computation, 183, 717-728.
11. Pandey, P., Punnen, A. P., (2007). A Simplex Algorithm for Piecewise-Linear Fractional Programming Problems. European Journal of Operational Research, 178, 343-358.
12. Tantawy, S. F., (2008). A New Procedure for Solving Linear Fractional Programming Problems. Mathematical and Computer Modelling, 48, 969-973.
13. Effati, S., Pakdaman, M., (2012). Solving the Interval-Valued Linear Fractional Programming Problem. American Journal of Computational Mathematics, 2, 51-55.
14. Mehra, A., Chandra, S., Bector, C. R., (2007). Acceptable Optimality in Linear Fractional Programming with Fuzzy Coefficients. Fuzzy Optimization and Decision Making, 6(1): 5–16.
15. Zhu, H., Huang, G. H., (2011). A Stochastic Linear Fractional Programming Approach for Sustainable Waste Management. Waste Management, 31(12), 2612–2619.