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Optimization of Solution Regularized Long-Wave Equation by Using Modified Variational Iteration Method

T. Allahviranloo*, Sh. Sadigh Behzadi

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Abstract In this paper, a regularized long-wave equation (RLWE) is solved by using the Adomian's decomposition method (ADM), modified Adomian's decomposition method (MADM), variational iteration method (VIM), modified variational iteration method (MVIM), and homotopy analysis method (HAM). The approximate solution of this equation is calculated in the form of series whose components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed methods are proved. A numerical example is studied to demonstrate the accuracy of the presented methods.

Keywords: Regularized Long-Wave Equation, Adomian Decomposition Method, Modified Adomian Decomposition Method, Variational Iteriation Method, Modified Variational Iteration Method, Homotopy Analysis Method.

1 Introduction

The RLW equation has a higher order nonlinearity of the form

$$u_t + u_x + au^n u_x + u_{xx} = 0, n \geq 1 \tag{1}$$

where a is a constant. The case $n = 1$ corresponds to the RLW equation, which was first proposed in 1972 by Benjamin, et al. [1]. This equation is an alternative to the Korteweg-de Vries (KdV) equation and describes the unidirectional propagation of small-amplitude long waves on the surface of water in a channel. The RLW equation is well-known in physical applications. This equation models long wave in a nonlinear dispersive system. Their solutions exhibit definite soliton-like behavior that is not explainable by any known theory [2]. The RLW equation is $n = 2$ used in the analysis of the surface waves of long wavelength in liquids, of hydromagnetic waves in cold plasma, a coustic-gravity wave in compressible fluids and acoustic waves in anharmonic crystals, where $n = 2$, the RLW equation is called the modified RLW equation (mRLWE). A lot of works have been done in order to find the numerical solution of this equation. For example, [3-22],variational iteration method [23,-25], homotopy analysis method [26].

In this work, we develop the ADM, MADM, VIM, MVIM and HAM to solve the Eq.(1)

* **Corresponding Author.** (✉)
E-Mail: allahviranloo@yahoo.com (T. Allahviranloo)

T. Allahviranloo
Department of Mathematics, Parand Branch, Islamic Azad University, Parand, Iran.

Sh. Sadigh Behzadi
Department of Mathematics, Qazvin Branch, Islamic Azad University, Qazvin, Iran.

with the initial conditions:

$$u(x,0)=f(x), \tag{2}$$

$$u_{xx}(x,0)=g(x),$$

The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq. (1). In section 3 we prove the existence and uniqueness of the solution and convergence of the proposed methods. Finally, a numerical example is presented in section 4 to illustrate the accuracy of these methods.

In order to obtain an approximate solution of Eq. (1), let us integrate one time Eq. (1) with respect to t using the initial conditions we obtain,

$$u(x,t)=-F(x,t)-\int_0^t D^1(u''(x,t))dt-\int_0^t H(u(x,t))dt, \tag{3}$$

where,

$$D^i(u(x,t))=\frac{\partial^i u(x,t)}{\partial x^i}, \quad i=1,2,$$

$$F(x,t)=f(x)+g(x)+D^2(u(x,t)),$$

$$H(u(x,t))=au^n(x,t)D(u(x,t)).$$

In Eq. (3), we assume $F(x,t)$ is bounded for all x,t in $J=[0,T]\,(T\in R)$

The terms $D(u(x,t)),H(u(x,t))$ are Lipschitz continuous with,

$$|D(u)-D(u^*)|\le L_1|u-u^*|$$

$$|H(u)-H(u^*)|\le L_2|u-u^*|.$$

2 The iterative methods

2.1 description of the madm and adm

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu+Ru+Nu=g_1, \tag{4}$$

where $u(x,t)$ is the unknown function, L is the highest order derivative operator, which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g_1 is the source term. Applying the inverse operator L^{-1} to both sides of Eq.(4), and using the given conditions we obtain

$$u(x,t)=f_1(x)-L^{-1}(Ru)-L^{-1}(Nu) \tag{5}$$

where the function $f_1(x)$ represents the terms arising from integrating the source term g_1, The nonlinear operator $Nu=G_1(u)$ is decomposed as

$$G_1(u)=\sum_{n=0}^{\infty}A_n, \tag{6}$$

where $A_n, n\ge 0$ are the Adomian polynomials determined formally as follows:

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]\right]_{\lambda=0} \tag{7}$$

The first Adomian polynomials (introduced in [27,28,29]) are:

$$
\begin{aligned}
&A_0 = G_1(u_0),\\
&A_1 = u_1 G_1'(u_0),\\
&A_2 = u_2 G_1'(u_0) + \frac{1}{2!} u_1^2 G_1''(u_0),\\
&A_3 = u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!} u_1^3 G_1'''(u_0), \ldots
\end{aligned} \tag{8}
$$

2.1.1 Adomian decomposition method

The standard decomposition technique represents the solution of $u(x,t)$ in (4) as the following series,

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{9}$$

where, the components $u_0, u_1, \ldots$ which can be determined recursively

$$
\begin{aligned}
&u_0 = -F(x,t),\\
&u_1 = -\int_0^t A_0(x,t)\,dt - \int_0^t B_0(x,t)\,dt,\\
&\vdots\\
&u_{n+1} = -\int_0^t A_n(x,t)\,dt - \int_0^t B_n(x,t)\,dt, \quad n \ge 0
\end{aligned} \tag{10}
$$

Substituting (8) into (10) leads to the determination of the components of u.

2.1.2 The modified adomian decomposition method

The modified decomposition method was introduced by Wazwaz [30]. The modified forms was established on the assumption that the function $F(x,t)$ can be divided into two parts, namely $F_1(x,t)$ and $F_2(x,t)$. Under this assumption we set

$$F(x,t) = F_1(x,t) + F_2(x,t). \tag{11}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part F_1 be assigned to the zeroth component u_0, whereas the remaining part F_2 be combined with the other terms given in (11) to define u_1. Consequently, the modified recursive relation

$$
\begin{aligned}
&u_0 = -F_1(x,t),\\
&u_1 = -F_2(x,t) - L^{(-1)}(Ru_0) - L^{(-1)}(A_0),\\
&\vdots\\
&u_{n+1} = -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \ge 1,
\end{aligned} \tag{12}
$$

was developed.

To obtain the approximation solution of Eq. (1), according to the MADM, we can write the iterative formula (12) as follows:

$$\begin{aligned}
&u_0 = -F_1(x,t),\\
&u_1 = -F_2(x,t) - \int_0^t A_0(x,t)\,dt - \int_0^t B_0(x,t)\,dt,\\
&\vdots\\
&u_{n+1} = -\int_0^t A_n(x,t)\,dt - \int_0^t B_n(x,t)\,dt.
\end{aligned} \tag{13}$$

The operators $D(u), H(u)$ are usually represented by the infinite series of the Adomian polynomials as follows:

$$D(u) = \sum_{i=0}^{\infty} A_i,$$

$$H(u) = \sum_{i=0}^{\infty} B_i.$$

where A_i and B_i are the Adomian polynomials.

Also, we can use the following formula for the Adomian polynomials [31]:

$$\begin{aligned}
&A_n = D(s_n) - \sum_{i=0}^{n-1} A_i,\\
&B_n = H(s_n) - \sum_{i=0}^{n-1} B_i.
\end{aligned} \tag{14}$$

Where $s_n = \sum_{i=0}^{n} u_i(x,t)$ is the partial sum.

3 Description of the vim and mvim

In the VIM [32-33], it has been considered the following nonlinear differential equation:

$$Lu + Nu = g_1, \tag{15}$$

where L is a linear operator, N is a nonlinear operator, and g_1 is a known analytical function. In this case, the functions u_1 may be determined recursively by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,\tau)\{L(u_n(x,\tau)) + N(u_n(x,\tau)) - g_1(x,\tau)\}\,d\tau, n \geq 0, \tag{16}$$

where λ is a general lagrange multiplier, which can be computed using the variational theory. Here the function $u_n(x,t)$ is a restricted variations, which means $\delta u_n = 0$. Therefore, we first determine the lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(x,t), n \geq 0$, of the solution $u(x,\tau)$ will be readily obtained upon using the obtained lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximations $u_n(x,t), n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{17}$$

The VIM has been shown to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converge rapidly to accurate solutions.

To obtain the approximation solution of Eq. (1), according to the VIM, we can write iteration formula (16) as follows:

$$u_{n+1}(x,t) = u_n(x,t) + L^{-1}\left(\lambda\left[u_n(x,t) + F(x,t) + \int_0^t D(u_n(x,t))dt + \int_0^t H(u_n(x,t))dt\right]\right), n \geq 0 \tag{18}$$

where,

$$u_t^{-1}(.) = \int_0^t (.) d\tau$$

To find the optimal λ, we proceed as

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta L_t^{-1}\left(\lambda\left[u_n(x,t) + F(x,t) + \int_0^t D(u_n(x,t))dt + \int_0^t H(u_n(x,t))dt\right]\right). \tag{19}$$

From Eq. (19), the stationary conditions can be obtained as follows:

$\acute{\lambda} = 0$ and $1 + \lambda = 0$.

Therefore, the Lagrange multipliers can be identified as $\lambda = -1$ and by substituting in (18), the following iteration formula is obtained.

$$u_0 = -F(x,t),$$

$$u_{n+1}(x,t) = u_n(x,t) + L_t^{-1}\left(u_n(x,t) + F(x,t) + \int_0^t D(u_n(x,t))dt + \int_0^t H(u_n(x,t))dt\right), n \geq 0. \tag{20}$$

To obtain the approximation solution of Eq. (1), based on the MVIM [34,35,36], we can write the following iteration formula:

$$u_0 = -F(x,t),$$

$$u_{n+1}(x,t) = u_n(x,t) + L_t^{-1}\left(\int_0^t D(u_n(x,t) - u_{n-1}(x,t))dt + \int_0^t H(u_n(x,t) - u_{n-1}(x,t))dt\right), n \geq 0. \tag{21}$$

Relations (20) and (21) will enable us to determine the components $u_n(x,t)$ recursively for $n \geq 0$.

4 Description of the ham

Consider

$$N[u] = 0,$$

where N is a nonlinear operator, $u(x,t)$ is an unknown function and x is an independent variable. Let $u_0(x,t)$ denotes an initial guess of the exact solution $u(x,t), h \neq 0$, an auxiliary parameter, $H_1(x,t) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x,t)] = 0$, when $s(x,t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L\left[\Phi(x,t;q)-u_0(x,t)\right]-qhH_1(x,t)N\left[\Phi(x,t;q)\right]= \hat{H}\left[\Phi(x,t;q);u_0(x,t),H_1(x,t),h,q\right] \tag{22}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x,t)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H_1(x,t)$.

Enforcing the homotopy (22) to be zero, i.e.,

$$\hat{H}_1\left[\Phi(x,t;q);u_0(x,t),H_1(x,t),h,q\right]=0, \tag{23}$$

we have the so-called zero-order deformation equation

$$(1-q)L\left[\Phi(x,t;q)-u_0(x,t)\right]=qhH_1(x,t)N[\Phi(x,t;q)]. \tag{24}$$

When $q=0$, the zero-order deformation Eq. (24) becomes

$$\Phi(x;0)=u_0(x,t), \tag{25}$$

and when $q=1$, since $h\neq 0$ and $H_1(x,t)\neq 0$, the zero-order deformation Eq. (24) is equivalent to

$$\Phi(x,t;1)=u(x,t). \tag{26}$$

Thus, according to (25) and (26), as the embedding parameter q increases from $0\ to\ 1,\Phi(x,t;q)$, varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$. Such a kind of continuous variation is called deformation in homotopy [37,38].

Due to Taylor's theorem, $\Phi(x,t;q)$ can be expanded in a power series of q as follows

$$\Phi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m, \tag{27}$$

where

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\ \ \Phi(x,t;q)}{\partial q^m}\Big|_{q=0.}$$

Let the initial guess$u_0(x,t)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function$H_1(x,t)$ be properly chosen so that the power series (27) of $\Phi(x,t;q)$ converges at $q=1$, then, we have under these assumptions the solution series

$$u(x,t)=\Phi(x,t;1)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t). \tag{28}$$

From Eq. (27), we can write Eq. (24) as follows

$$(1-q)L\left[\Phi(x,t;q)-u_0(x,t)\right]=(1-q)L\left[\sum_{m=1}^{\infty}u_m(x,t)q^m\right]=qhH_1(x,t)N\left[\Phi(x,t;q)\right]\rightarrow L[\sum_{m=1}^{\infty}u_m(x,t)q^m]-q\,L\left[\sum_{m=1}^{\infty}u_m(x,t)q^m\right]=qhH_1(x,t)N\left[\Phi(x,t;q)\right] \tag{29}$$

By differentiating (29) m times with respect to q, we obtain

$$\{L[\sum_{m=1}^{\infty} u_m(x,t)q^m]-q\,L\left[\sum_{m=1}^{\infty} u_m(x,t)q^m\right]\}^{(m)}=\left\{qhH_1(x,t)N\left[\Phi(x,t;q)\right]\right\}^m=$$

$$m!L\left[u_m(x,t)-u_{m-1}(x,t)\right]=hH_1(x,t)m\frac{\partial^{m-1}\mathrm{N}\left[\Phi(x,t;q)\right]}{\partial q^{m-1}}|_{q=0}.$$

Therefore,

$$L\left[u_m(x,t)-\chi_\mu u_{m-1}(x,t)\right]=hH_1(x,t)R_m\left(u_{m-1}(x,t)\right), \tag{30}$$

where,

$$R_m\left(u_{m-1}(x,t)\right)=\frac{1}{(m-1)!}\frac{\partial^{m-1}N\left[\Phi(x,t;q)\right]}{\partial q^{m-1}}|_{q=0,} \tag{31}$$

and

$$x_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}$$

Note that the high-order deformation Eq. (30) is governing the linear operator$L$, and the term can be $R_m\left(u_{m-1}(x,t)\right)$ expressed simply by (31) for any nonlinear operator N.

To obtain the approximation solution of Eq. (1), according to HAM, let

$$N\left[u(x,t)\right]=u(x,t)+F(x,t)+\int_0^t D\left(u(x,t)\right)dt+\int_0^t H\left(u(x,t)\right)dt,$$

so,

$$R_m\left(u_{m-1}(x,t)\right)=u_{m-1}(x,t)+F(x,t)+\int_0^t D\left(u_{m-1}(x,t)\right)dt+\int_0^t H\left(u_{m-1}(x,t)\right)dt, \tag{32}$$

Substituting (32) into (30)

$$\begin{aligned}L\left[u_m(x,t)-\chi_\mu u_{m-1}(x,t)\right]&=hH_1(x,t)[u_{m-1}(x,t)+\int_0^t D\left(u(x,t)\right)dt+\\ &\int_0^t H\left(u(x,t)\right)dt+(1-\chi_m)F(x,t)].\end{aligned} \tag{33}$$

We take an initial guess$u_0(x,t)=-F(x,t)$, an auxiliary linear operator $Lu=u$, a nonzero auxiliary parameter $h=-1$, and auxiliary function $H_1(x,t)=1$. This is substituted into (33) to give the recurrence relation

$$\begin{aligned}&u_0(x,t)=-F(x,t),.5cm\\ &u_{n+1}(x,t)=-\int_0^t D\left(u_n(x,t)\right)dt-\int_0^t H\left(u_n(x,t)\right)dt,\quad n\ge 1.\end{aligned} \tag{34}$$

Therefore, the solution$u(x,t)$ becomes

$$u(x,t)=\sum_{\mathrm{n}=0}^{\infty} u_n(x,t)=-F(x,t)+\sum_{n=1}^{\infty}\left(-\int_0^t D\left(u_n(x,t)\right)dt-\int_0^t H\left(u_n(x,t)\right)dt\right). \tag{35}$$

which is the method of successive approximations. If

$\left|u_n(x,t)\right|<1,$

then the series solution (35) convergence uniformly.

5 Existence and convergency of iterative methods

We set,

$\alpha := T\left(L_1 + L_2\right),$

$\beta := 1 - T\left(1-\alpha\right), \qquad \gamma := 1 - TL\alpha.$

Theorem 3.1
Let $0 < \alpha < 1$, then RLW equation has a unique solution.
Proof
Let u and u^* be two different solutions of (3); then

$$\left|u - u^*\right| = \left|-\int_0^t D\left(u\left(x,t\right)\right)dt - \int_0^t H\left(u\left(x,t\right)\right)dt\right|$$

$$\leq \int_0^t \left|D\left(u\left(x,t\right)\right) - D\left(u^*\left(x,t\right)\right)\right|dt + \int_0^t \left|H\left(u\left(x,t\right)\right) - H\left(u^*\left(x,t\right)\right)\right|dt$$

$$\leq T\left(L_1 + L_2\right)\left|u - u^*\right| = \alpha\left|u - u^*\right|.$$

From which we get $(1-\alpha)\left|u - u^*\right| \leq 0$. Since $0 < \alpha < 1$, then $|u - u^*| = 0$. Implies $u = u^*$ and completes the proof.

Theorem 3.2
The series solution $u\left(x,t\right) = \sum_{i=0}^{\infty} u_i\left(x,t\right)$ of problem (1) using MADM convergence when $0 < \alpha < 1$, $\left|u_1\left(x,t\right)\right| < \infty$.
Proof
Denote as $(C[J], \|.\|)$ the Banach space of all continuous functions on J with the norm, for all $\left\|f\left(t\right)\right\| = max\left|f\left(t\right)\right|$ t in J. Define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n \geq m$. We are going to prove that s_n is a Cauchy sequence in this Banach space:

$$\left\|s_n - s_m\right\| = max_{\forall t \in J}\left|s_n - s_m\right| = max_{\forall t \in J}\left|\sum_{i=m+1}^{n} u_i\left(x,t\right)\right|$$

$$= max_{\forall t \in J}\left|\sum_{i=m+1}^{n}\left(-\int_0^t A_{i-1}dt - \int_0^t B_{i-1}dt\right)\right| = max_{\forall t \in J}\left|-\int_0^t\left(\sum_{i=m}^{n-1} A_i\right)dt - \int_0^t \sum_{i=m}^{n-1} B_i dt\right|.$$

From [6], we have

$$\sum_{i=m}^{n-1} A_i = D\left(s_{n-1}\right) - D\left(s_{m-1}\right),$$

$$\sum_{i=m}^{n-1} B_i = H\left(s_{n-1}\right) - H\left(s_{m-1}\right).$$

So,

$$\|s_n - s_m\| = max_{\forall t \in J} \left| -\int_0^t \left[D\left(s_{n-1}\right) - D\left(s_{m-1}\right)\right] dt - \int_0^t \left[H\left(s_{n-1}\right) - H\left(s_{m-1}\right)\right] dt \right|$$
$$\leq \int_0^t \left|D\left(s_{n-1}\right) - D\left(s_{m-1}\right)\right| dt - \int_0^t \left|H\left(s_{n-1}\right) - H\left(s_{m-1}\right)\right| dt$$
$$\leq \alpha \|s_n - s_m\|.$$

Let $n = m+1$; then
$\|s_n - s_m\| \leq \alpha \|s_m - s_{m-1}\| \leq \alpha^2 \|s_{m-1} - s_{m-2}\| \leq \cdots \leq \alpha^m \|s_1 - s_0\|$
From the triangle inequality we have
$$\|s_n - s_m\| \leq \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \cdots + \|s_n - s_{n-1}\| \leq \left[\alpha^m + \alpha^{m+1} + \ldots + \alpha^{n-m-1}\right] \|s_1 - s_0\|$$
$$\leq \alpha^m \left[1 + \alpha + \alpha^2 + \ldots + a^{n-m-1}\right] \|s_1 - s_0\|$$
$$\leq \alpha^m \left[\frac{1 - a^{n-m}}{1-\alpha}\right] \|u_1(x,t)\|.$$
Since $0 < \alpha < 1$, we have $(1 - a^{n-m}) < 1$, then
$$\|s_n - s_m\| \leq \frac{\alpha^m}{1-\alpha} max_{\forall t \in J} |u_1(x,t)|.$$
But $|u_1(x,t)| < \infty$, so, as $m \to \infty$, then $\|s_n - s_m\| \to 0$. We conclude that s_n is a Cauchy sequence in $C[J]$; therefore, the series is convergence and the proof is complete.

Theorem 3.3
The solution $u_n(x,t)$ obtained from the relation (20) using VIM converges to the exact solution of the problem (1) whe $0 < \alpha < 1$, n and $0 < \beta < 1$.
Proof
$$u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}\left(\left[u_n(x,t) + F(x,t) + \int_0^t D\left(u_n(x,t)\right) dt + \int_0^t H\left(u_n(x,t)\right)) dt\right]\right) \quad (36)$$
$$u(x,t) = u(x,t) - L_t^{-1}([u(x,t) + F(x,t) \int_0^t D\left(u(x,t)\right) dt + \int_0^t H\left(u(x,t)\right)) dt \quad (37)$$
By subtracting relation (36) from (37),
$$u_{n+1}(x,t) - u(x,t) = u_n(x,t) - u(x,t) - L_t^{-1}\left(u_n(x,t) - u(x,t) - \int_0^t \left[D\left(u_n(x,t)\right) - D\left(u(x,t)\right)\right] dt - \int_0^t \left[H\left(u_n(x,t)\right) - H\left(u(x,t)\right)\right] dt) dt\right),$$
if we set, $e_{n+1}(x,t) = u_{n+1}(x,t) - u_n(x,t)$, $e_n(x,t) = u_n(x,t) - u_n(x,t)$,
$|e_n(x,t^*)| = max_t |e_n(x,t)|$ then since e_n is a decreasing function with respect to t from the mean value theorem we can write,

$$e_{n+1}(x,t)=e_n(x,t)+L_t^{-1}(e_n(x,t)+\int_0^t\left[D\left(u_n(x,t)\right)-D\left(u(x,t)\right)\right]dt$$

$$+\int_0^t\left[H\left(u_n(x,t)\right)-H\left(u(x,t)\right)\right]dt)$$

$$\leq e_n(x,t)+L_t^{-1}(-e_n(x,t)+L_t^{-1}\left|e_n(x,t)\right|T\left(L_1+L_2\right))$$

$$\leq e_n(x,t)-Te_n(x,\eta)+T\left(L_1+L_2\right)L_t^{-1}L_t^{-1}\left|e_n(x,t)\right|)$$

$$\leq 1-T\left(1-\alpha\right)\left|e_n\left(x,t^*\right)\right|,$$

where $0\leq\eta\leq t$. Hence, $e_{n+1}(x,t)\leq\beta\left|e_n\left(x,t^*\right)\right|$.Therefore,

$$\left\|e_{n+1}\right\|=max_{\forall t\in J}\left|e_{n+1}\right|\leq\beta max_{\forall t\in J}\left|e_n\right|\leq\beta\left\|e_n\right\|.$$

Since $0<\beta<1$, then $e_n\rightarrow 0$. So the series converges and the proof is complete.

Theorem 3.4

The solution $u_n(x,t)$ obtained from the relation (21) using MVIM for the problem (1) converges when.

$0<\alpha<1, 0<\gamma<1$.

Proof

The Proof is similar to the previous theorem.

Theorem 3.5

If the series solution (34) of problem (1) using HAM is convergent, then it converges to the exact solution of the problem (1).

Proof

We assume:

$$u(x,t)=\sum_{m=0}^{\infty}u_m(x,t),$$

$$\hat{D}\left(u(x,t)\right)=\sum_{m=0}^{\infty}D\left(u_m(x,t)\right),$$

$$\hat{H}\left(u(x,t)\right)=\sum_{m=0}^{\infty}H\left(u_m(x,t)\right).$$

where,

$$\lim_{m\rightarrow\infty}u_m(x,t)=0.$$

We can write,

$$\sum_{m=1}^{n}\left[u_m(x,t)-\chi u_{m-1}(x,t)\right]=u_1+\left(u_2-u_1\right)+\ldots+\left(u_n-u_{n-1}\right)=u_n(x,t). \tag{38}$$

Hence, from (38),

$$\lim_{n\rightarrow\infty}u_n(x,t)=0 \tag{39}$$

So using (39) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L\left[u_m(x,t)-\chi u_{m-1}(x,t)\right]=L[\sum_{m=1}^{\infty}\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]]=0.$$

therefore, from (30), we can obtain that,

$$\sum_{m=1}^{\infty} L\left[u_m(x,t)-\chi u_{m-1}(x,t)\right]=hH_1(x,t)\sum_{m=1}^{\infty} R_{m-1}\left(u_{m-1}(x,t)\right)=0.$$

Since $h \neq 0$ and $H_1(x,t)\neq 0$, we have

$$\sum_{m=1}^{\infty} R_{m-1}\left(u_{m-1}(x,t)\right)=0 \tag{40}$$

By substituting $\sum_{m=1}^{\infty} R_{m-1}\left(u_{m-1}(x,t)\right)$ into the relation (40) and simplifying it, we have

$$\begin{aligned}\sum_{m=1}^{\infty} R_{m-1}\left(u_{m-1}(x,t)\right)&=\sum_{m=1}^{\infty}[u_{m-1}(x,t)+\int_0^t D\left(u_{m-1}(x,t)\right)dt+\int_0^t H\left(u_{m-1}(x,t)\right)dt\\&+(1-\chi_m)F(x,t)]=u(x,t)+F(x,t)+\int_0^t \hat{D}\left(u(x,t)\right)dt+\int_0^t \hat{H}\left(u(x,t)\right)dt.\end{aligned} \tag{41}$$

From (40) and (41), we have

$$u(x,t)=-F(x,t)-\int_0^t \hat{D}\left(u(x,t)\right)dt-\int_0^t \hat{H}\left(u(x,t)\right)dt,$$

therefore, $u(x,t)$ must be the exact solution.

6 Numerical example

In this section, we compute a numerical example, which is solved by the ADM, MADM, VIM, MVIM, and HAM. The program has been provided with Mathematica 6 according to the following algorithm, where ε is a given positive value.

Algorithm: (ADM, MADM and HAM)

Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relations (10) for ADM, (13) for MADM and (34) for HAM.
Step 3. If $|u_{n+1}-u_n|<\varepsilon$; then go to step 4,
else $n \leftarrow n+1$, and go to step 2.
Step 4. Print $u(x,t)=\sum_{i=0}^{n} u_i(x,t)$ as the approximate of the exact solution.

Algorithm: (VIM and MVIM)

Step 1. Set $\leftarrow 0$.
Step 2. Calculate the recursive relations (20) for VIM and (21) for MVIM.
Step 3. If$|u_{n+1}-u_n|<\varepsilon$; then go to step 4,
else $n \leftarrow n+1$, and go to step 2.
Step 4. Print $u_n(x,t)$ as the approximate of the exact solution.

Example

Consider the RLW equation as follows:

$u_t^6 - u_x xx^4 + 6u_x^4 = 0$

subject to the initial conditions:

$u(x,0) = \cos^{\frac{1}{2}}(x)$

with the exact solution is $u(x,t) = \cos^{\frac{1}{2}}(x - t)$.

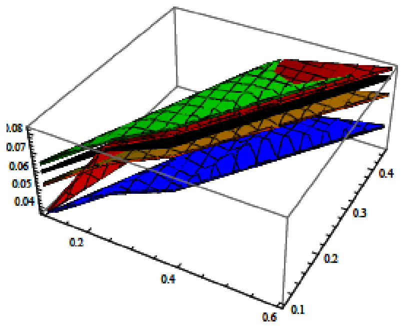

Fig. 1 The Comparison between the results of the methods in the example 4.1, Green = Error ADM (n=11), Red = Error MADM (n=9), Black = Error VIM (n=6), Blue = Error MVIM (n=3), Orange= Error HAM (n=5).

Figure 1, shows that, approximate solution of the RLW equation is convergent with 3 iterations by using the MVIM. By comparing the results of Figure 1, we can observe that the MVIM has more rapid convergence than the ADM, MADM, VIM and HAM.

7 Conclusion

The MVIM has been shown to solve effectively, easily, and accurately a large class of nonlinear problems with the approximations whose convergence are rapidly to exact solutions. In this work, the MVIM has been successfully employed to obtain the approximate analytical solution of the RLW equation.

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