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An effective method based on the angular constraint to detect Pareto points in bi-criteria optimization problems

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Abstract The most important issue in multi-objective optimization problems is to determine the Pareto points along the Pareto frontier. If the optimization problem involves multiple conflicting objectives, the results obtained from the Pareto-optimality will have the trade-off solutions that shaping the Pareto frontier. Each of these solutions lies at the boundary of the Pareto frontier, such that the improvement in one of the objectives results in the worsening of at least one of the other objectives. Usually, it is not economical to generate the entire Pareto surface due to the high computational cost for function evaluations. Therefore, it is important to get a uniform distribution at the Pareto points in the Pareto frontier.

In this paper, an efficient method based on angular constraint is presented for finding a suitable approximation of the Pareto front of bi-objective optimization problems. In order to get a better distribution of points at the Pareto front it is used a strategy following closer the main shape of the frontier. To get only the global Pareto points, this strategy sweeps the objective space just once, getting automatically rid-off the non-Pareto and local Pareto points, without any further filtering. The researcher, after proposing an algorithm for the operation of the method, compares its efficiency in one test problem with methods such as weighted sum (WS) and epsilon-constraint methods. The obtained experimental results show that the proposed method is efficient and in most situations more accurate.

Keywords: Multi-objective Optimization, Pareto Optimality Concept, Scalarization Methods.

1 Introduction

Multi-objective optimization is concerned with mathematical optimization problems involving more than one objective function to be optimized simultaneously.

Multi-objective optimization has been applied in many fields of science, where optimal

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decisions need to be taken to the presence of trade-offs between two or more conflicting objectives [1-3]. For a nontrivial multi-objective optimization problem, no single solution exists that simultaneously optimizes each objective. In that case, the objective functions are said to be conflicting, and there exists a (possibly infinite) number of Pareto optimal solutions. A solution is called non-dominated if none of the objective functions can be improved in value without degrading some of the other objective values [4]. Without additional subjective preference information, all Pareto optimal solutions are considered equally good (as vectors cannot be ordered completely). Researchers study multi-objective optimization problems (MOPs) from different viewpoints and, thus, there exist different solution philosophies and goals when setting and solving them. The goal may be to find a representative set of Pareto optimal solutions, and/or quantify the trade-offs in satisfying the different objectives, and/or finding a single solution that satisfies the subjective preferences of a human decision maker (DM) [5]. There are generally two categories of MOP approaches, gradient-based methods and metaheuristic (nongradient-based) methods [6-8]. Gradient-based method usually conducts multiple single-objective optimizations, while metaheuristic methods can solve one single but larger MOP problem correspondingly. The simplest gradient-based MOP method is the weighted sum (WS) method [9, 10]. Another classical standpoint is the ε-constraint method [11] which is able to find the non-convex portion of the Pareto front, but the solution depends a lot on the ε vector. The ε-constraint method optimizes only one objective and transforms the other objectives into constraints when searching for Pareto solutions. Unlike metaheuristic method, the gradient-based methods require the gradient information of the functions, thus the functions should be continuous and differentiable. For some methods, such as sequential quadratic programming, the functions should be twice differentiable. There has been a great deal of effort by researchers in this area in recent years for developing methods to generate an approximation of the Pareto front [12-14].

The present idea proposes a dissimilar strategy in order to improve the efficiency and the distribution of the solutions relatively to the methods just described. The proposed approach stems basically in substituting the scalar product constraint by a constraint formulated throughout the ray with origin at the point (0,0) of the normalized criterion space (see Fig.1).

This approach, no Pareto-filtering is necessary and only is enough the search from right to left of the scalar optimization problems. Besides that, since the rays originated at the point (0,0) go along better with the global curvature of the Pareto frontier than the normal directions to the ideal line, then we get a better distribution of solutions along that frontier.

The rest of this paper is organized as follows. Section 2 provides the necessary background and terminology used in this paper. Section 3 presents the detailed procedure and discussion of the proposed angular constraint method (ACM). Section 4 demonstrates the applicability and efficiency of the approaches using one numerical example. Finally, conclusions are presented in Section 5.

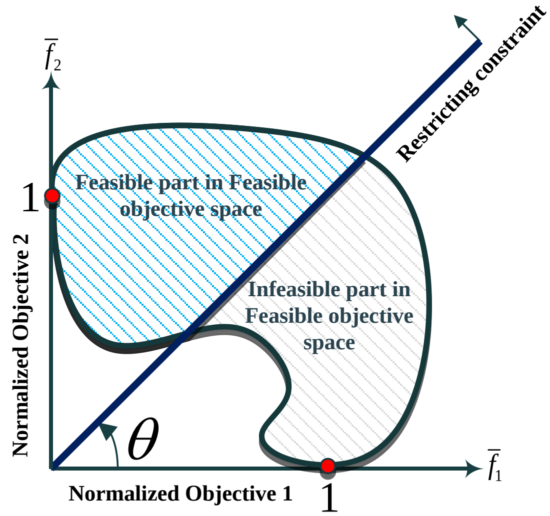

Fig.1 Schematic of proposed method

2 Backgrounds and terminology

A typical multi-objective optimization problem (MOP) can be formulated as in the following equation:

$$\begin{aligned} &\min \ f(x) = \left(f_1(x), \cdots, f_p(x) \right), \ p \geq 2 \\ &s.t. \\ &\quad x \in X \end{aligned} \tag{1}$$

where $X \subseteq R^n$ is a nonempty feasible set, and $f(x)$ represents the vector of objectives and $f_k, k = 1, \cdots, p \ (p \geq 2)$ is a scalar decision variable which maps decision variable x into the objective space $f_k : R^n \to R$. The image of X under f is denoted by $Y := f(x) \subseteq R^p$ and referred to as the image space. When the objective functions conflict with each other, no single solutions can simultaneously minimize all scalar objective functions $f_k(x), k = 1, \cdots, p$. Therefore, it is necessary to introduce a new notion of optimality or Pareto efficiency, which is an important criterion for evaluating economic and engineering systems.
For $y, \tilde{y} \in R^p$,

- $y < \tilde{y}$ means $y_k < \tilde{y}_k$ for all $k = 1, \cdots, p$,
- $y \leqq \tilde{y}$ means $y_k \leq \tilde{y}_k$ for all $k = 1, \cdots, p$,
- $y \leq \tilde{y}$ means $y \leqq \tilde{y}$ but $y \neq \tilde{y}$.

In this paper, we use the above componentwise orders to order the objective space and define the cone $R^p_{\geqq} = \left\{ x \in R^p \,\middle|\, x \geqq 0_p \right\}$.

Definition 1.1 A feasible point $x^* \in X$ is called

- A weakly efficient solution of MOP (1), if there is no other $x \in X$ such that $f(x) < f(x^*)$. If $x^* \in X$ is weakly efficient then $f(x^*)$ is called a weakly non-dominated point.
- An efficient solution of MOP (1), if there is no other $x \in X$ such that $f(x) \leq f(x^*)$. If $x^* \in X$ is efficient then $f(x^*)$ is called a non-dominated point.

The set of all weakly efficient and efficient solutions of MOP (1) are denoted by X_{wE} and X_E, respectively. We call this image, weakly non-dominated solutions and non-dominated solutions which are denoted by Y_{wN} and Y_N, respectively. The goal of MOP is to identify Y_{wN} or Y_N which is able to represent the Pareto front.

Definition 1.2 For a p-objective problem, the point $f^{i*} = \left(f_1(x_i^*), \cdots, \mathrm{f}_p(x_i^*) \right)^t$ in which $x_i^* = \arg\min_{x \in X} f_i(x), i = 1, \cdots, p,$ is called the i-th anchor point. The anchor point in the feasible objective space corresponds to the best possible values for respective individual objectives.

Definition 1.3 The point $f^I = \left(f_1(x_1^*), \cdots, f_p(x_p^*) \right)^t$ in which $x_i^* = \arg\min_{x \in X} f_i(x), i = 1, \cdots, p,$ is called the ideal point of MOP (1).

To this end, we assume that MOP (1) has the individual ideal point.

Definition 1.4 Nadir point is a point in the criterion space where all objectives are simultaneously at their worst values. The nadir point is written as $f^N = \left(f_1^N, \cdots, f_p^N \right)^t$ where f_i^N is defined as $f_i^N = \max_{x \in X} f_i(x)$.

Since the values of the criteria components may differ greatly, they are normalized as

$$\overline{f}_k := \frac{f_k - f_k^I}{f_k^N - f_k^I}, k = 1, \cdots, p. \tag{2}$$

where f_k^I and f_k^N are k-th element of ideal and nadir points, respectively.

This way, the values of the normalized criteria components range from 0 to 1. Therefore, for bi-objective optimization problems the anchor points of the normalized criterion space are (0,1) and (1,0), respectively. Also, the ideal point in this space is the point (0,0).

3 Description of ACM method

Determining a uniform distribution of Pareto points is the main idea of this article. In order to achieve this goal, we use a different technique. This technique is based on limiting the normalized criterion space. By introducing a special linear limitation, we divide the normalized criterion space into two parts (see Fig.1). This causes the criterion space to be divided into two sections of the feasible part and infeasible part, respectively. We split the criterion space in two regions, for each scalar optimization problem, by using the ray with origin at the space. The gradient of this ray is as follows.

$$m = \tan\theta,\ 0 \le \theta \le \frac{\pi}{2} \tag{3}$$

If the number of angular segments is n, then the segment size will be equal to $\frac{\pi}{2n}$. Hence the criterion space split by the line $\bar{f}_2 = \bar{f}_1 \cdot \tan\theta_i$ where $\theta_i = \frac{\pi}{2n} i$ for $i = 0,1,\cdots,n$ and the feasible region will be the region above this line. Since this line goes better with the global curvature of the feasible space boundary, we may obtain a better distribution of solutions.

Then the set $\Lambda = \left\{ \left(\bar{f}_1, \bar{f}_2 \right) \middle| \bar{f}_2 = \bar{f}_1 \cdot \tan\theta_i,\ \bar{f}_1 \ge 0,\ \bar{f}_2 \ge 0 \right\}$ is defined. It is clear that, we will use from this set as a starting point for achieving the Pareto frontier. The geometric of the Λ is shown in Fig.2.

Fig.2 illustrated a description of the ACM method for a two objective problem. Thus, the optimization problem must be solved to generate the Pareto frontier as:

$$\begin{aligned} &\min\ \ \bar{f}_2 \\ &s.t. \\ &\qquad \bar{f} \in \Lambda \cap f(X), \\ &\qquad x \in X \end{aligned} \tag{4}$$

According to problem (4), a point $\bar{x}_i \in X$ is efficient solution if

$$\begin{aligned} &\bar{x}_i := \arg\min\ \ \bar{f}_2 \\ &s.t. \\ &\qquad \bar{f}_2 - \bar{f}_1 \cdot \tan\frac{\pi i}{2n} \ge 0, i = 1,\cdots,n-1 \\ &\qquad \bar{f}_1^{\,i-1} - \bar{f}_1^{\,i} \ge 0, i = 1,\cdots,n-1 \\ &\qquad x \in X, \\ &\qquad \bar{f}_1, \bar{f}_2 \ge 0 \end{aligned} \tag{5}$$

Points in non-convexities regions of normalized criterion space are automatically eliminated if we use the angular constraint. Also, points in non-convexities are automatically filtered by the constraint $\overline{f}_1^{\,i-1} - \overline{f}_1^{\,i} \geq 0$ for $i = 1, \cdots, n-1$.

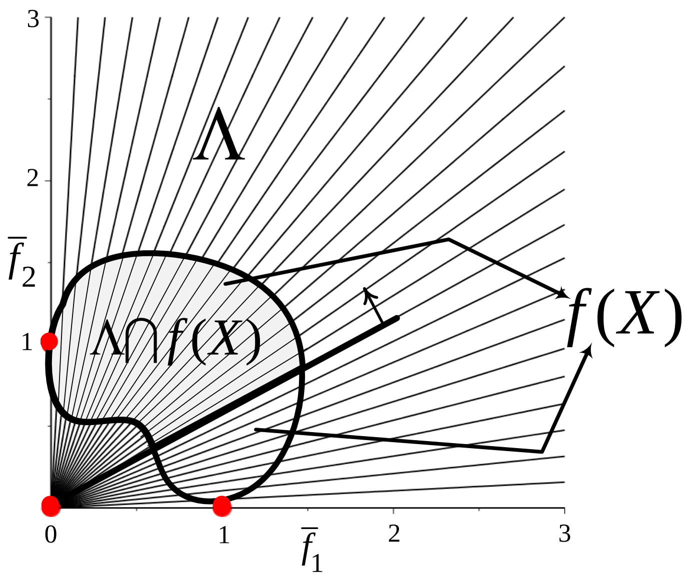

Fig.2 The sets of $f(x)$, Λ and its intersection with $\theta = \frac{\pi}{60}$ for normalized two objective

In the next section, we demonstrate by using one example that the ACM method is more efficient than the weighted sum method (WS) and ε-constraint method. It is notable that all single objective optimization problems in this paper are solved using the GlobalSolve command in MAPLE v.2016 with the active-set option.

4 Numerical simulation

In this section, the test problem modified ZDT3 is considered from [15].

4.1 Modified ZDT3 test problem

The modified ZDT3 test problem can be stated as follows:

$$
\begin{aligned}
&\min \quad f_1(x_1, \cdots, x_m) = x_1, \\
&\min \quad f_2(x_1, \cdots, x_m) = g(x_1, \cdots, x_m)\left(1 - \sqrt{\frac{f_1(x_1, \cdots, x_m)}{g(x_1, \cdots, x_m)}} - \frac{f_1(x_1, \cdots, x_m)}{g(x_1, \cdots, x_m)}\sin(10\pi x_1)\right), \\
&s.t. \\
&\qquad g(x_1, \cdots, x_m) = 1 + \frac{9}{m-1}\sum_{i=2}^{m} x_i^2, \\
&\qquad x_1 \in [0,1],\ x_i \in [-1,1], i = 2, 3, \ldots, m.
\end{aligned}
\tag{6}
$$

This is an $m = 30$ variable problem having a number of disconnected Pareto optimal fronts. The

Pareto optimal region corresponds to $0 \le x_1^* \le 1$ and $x_i^* = 0$ for $i = 2, 3, \cdots, 30$. The difficulty with this problem is that the Pareto optimal region is disconnected. The problem is solved by the ACM, the WS and ε-constraint methods with $n = 50$. The convergence of the Pareto front and distribution of solutions obtained by the ACM, WS and ε-constraint methods for finding 51 Pareto points after 30038040, 5522335 and 3356359 total function evaluations for this problem are depicted in Figs.3-5. Details are given in Table 1.

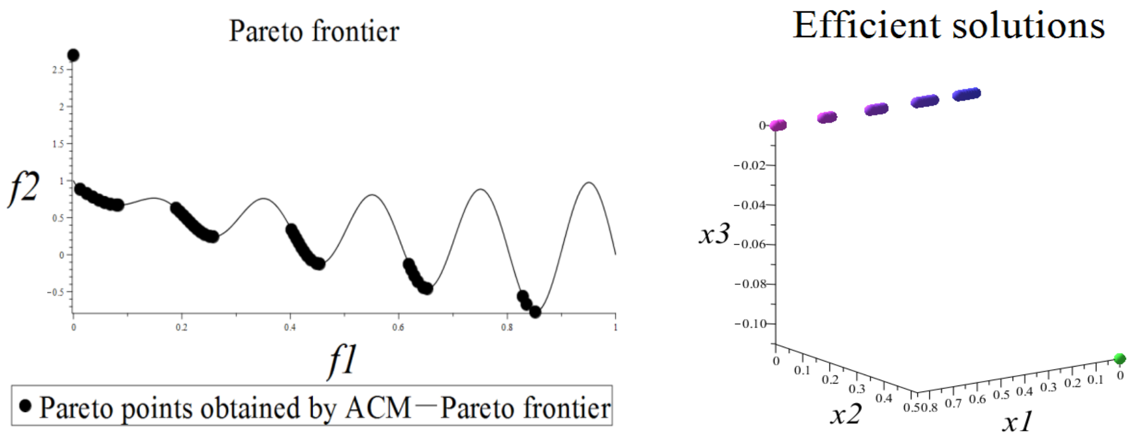

Fig.3 Pareto frontier and efficient solutions obtained by ACM with 30038040 total function evaluations

By a comparison of generational distance (GD), spread (SP) and extension (EX) (see [16]) for the three methods, it is concluded that the quality of the approximation of the Pareto optimal set, quality of the uniformity of the Pareto optimal set and quality of the well-extended of the optimal set obtained by the ACM is better than that for the WS and ε-constraint methods. A comparison of Figs.3-5 and Table 1 shows that the quality measure of the GD, SP and EX obtained by the ACM approach is better than the WS and ε-constraint methods.

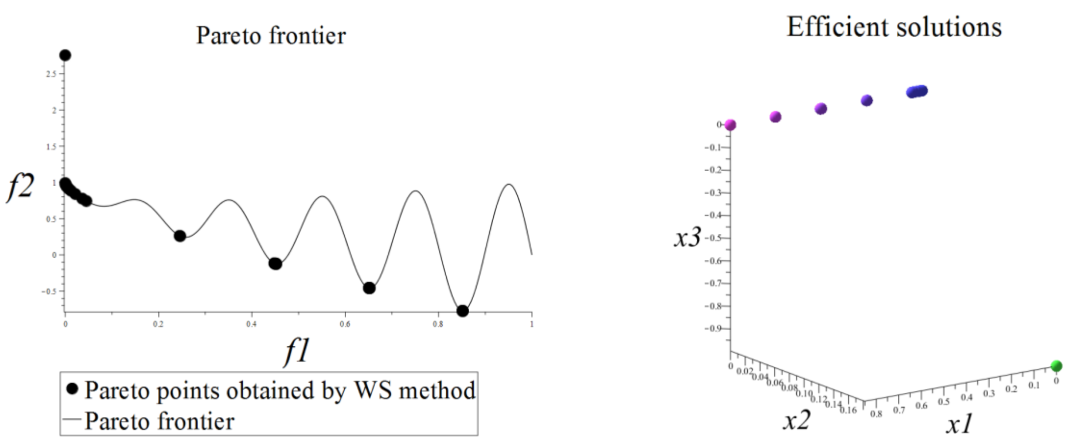

Fig.4 Pareto frontier and efficient solutions obtained by WS method with 5522335 total function evaluations

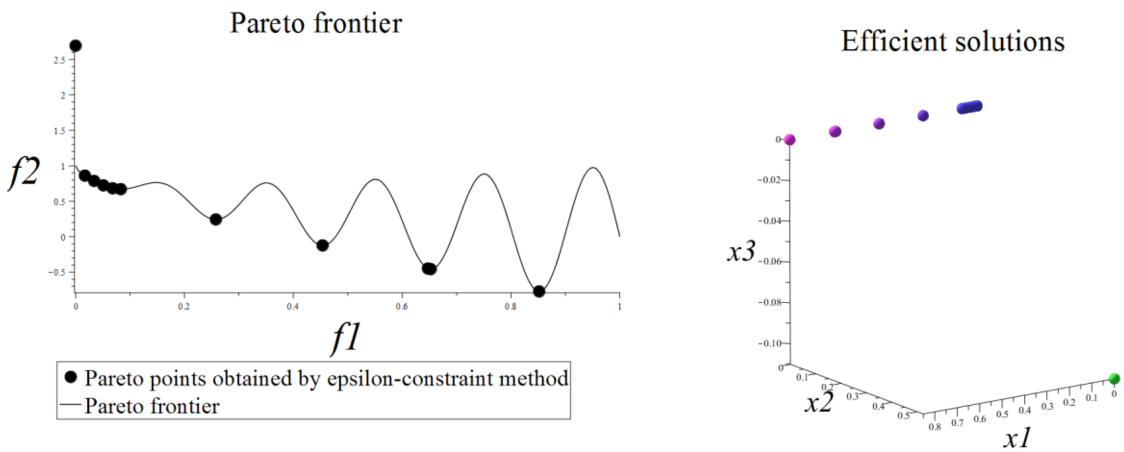

Fig.5 Pareto frontier and efficient solutions obtained by ε -constraint method with 3356359 total function evaluations

Table 1 Run time (s), total number of function evaluations (TFE), measure for the approximation (GD), measure of uniformity (SP) and measure of well-extended (EX) between solutions obtained by ACM, WS and ε -constraint methods

Method	**Modified ZDT3 test problem**				
	Run time (s)	**TFE**	**GD***	**SP***	**EX***
ACM	7483094	30038040	0.00009536	6.989	0.1714
WS	698390	5522335	0.001488	218.0	0.2180
ε -constraint	429687	3356359	0.003446	15.16	0.2334
* The numbers are rounded up to four significant digits.					

5 Conclusions

This article proposes a new strategy to calculate the global Pareto solutions for a bi-objective optimization problem. The method can calculate the scalar minimization sub-problems in only one-direction series and filtering automatically all the local Pareto and non-Pareto solutions, this way increasing the efficiency relatively to other strategies as, namely, the weighted sum and ε -constraint methods. Also, the method presented may obtain a better distribution of the Pareto points, since it is based on the division of the criterion space by rays originated at the right upper corner and directed nearly perpendicular to the Pareto frontier. The ACM approach generates a good distribution of the entire Pareto frontier for both the convex and non-convex Pareto front. The proposed method was applied to a test problem and performed very well for obtaining the Pareto front. In all cases, the solutions obtained by the ACM approach were better than the solutions obtained by the WS and ε -constraint methods.

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