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Some new concepts about IT2FNs with their usage in group decision-making problems

A. Keikha, H. Mishmast Nehi*

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Abstract Interavl type-2 fuzzy numbers (IT2FNs) are used in many real problems such as multiple attribute decision making (MADM) problems, to model the opinions/judgments of experts. This paper, using centroid points and uncertainty degrees of IT2FNs, presents a new method to rank them. Also, we present new methods based on Choquet integral and various types of Power average to aggregate a set of IT2FNs into a single one, separately. Finally, a new way is suggested to determine the importance degree of each criterion or decision maker in decision making problems. As an application, these methods will be applied to solve a group decision making problem.

Keywords: Multi-Attribute Decision Making, Choquet Integral, Ranking Function, IT2FN, Fuzzy Measure.

1 Introduction

Vagueness, ambiguity, and uncertainty are inseparable part of information which is resulted from human judgments or expressed by linguistic terms. Use of such information in many sciences such as Control, Robotic, Decision Making, etc., requires scientific modeling. For example, type-1 and type-2 Fuzzy Sets (T1FSs,T2FSs) [1, 2] have been proposed to model uncertain data. Because of computational complexity of T2FSs the authors are interested to apply Interval T2FSs (IT2FSs), i.e. a simplified form of T2FSs in which, their membership grades are subintervals from [0,1]. To date, there are many studies on T2FSs and IT2FSs theories [3-6, 7, 8, 9]. KM Algorithm proposed by Karnik and Mendel [10], to compute the centroid of an IT2FS. The centroid of an IT2FS measures the uncertainty, its properties presented in [5]. Wu et al. [11] proposed another method to compute the centroid of a T2FS. Set operations of T2FSs, i.e. union, intersection and, complement, without using the Extension Principle has been introduced by Mendel and John [12]. Cardinality, fuzziness, variance and skewness are four characteristic of IT2FSs, which are proposed by Mendel and Wu [3,9], to measure their uncertainty. Uncertainty degree of symmetric IT2FSs [13], the computation of all different uncertainty measures [14], α-cuts and α-planes of T2FSs [15, 8, 16], arithmetic operations between the trapezoidal IT2FSs [17, 18], ranking order of IT2FNs

* **Corresponding Author. (✉)**
E-mail: hmnehi@hamoon.usb.ac.ir (H. Mishmast Nehi)

A. Keikha
Ph.D of Applied Mathematics, Zahedan, Iran

H. Mishmast Nehi
Department of Mathematics, University of Siastan and Baluchestan, Zahedan, Iran

and their applications in solving fuzzy multiple attributes group decision making (FMAGDM) problems [17, 18, 19-21, 22] are some other researches that have been done about T2FSs.

Choquet integral (CI) as a more useful aggregation function, is applied to solve MADM problems [23, 24, 25, 26]. For example, to aggregate IT2FNs by CI, based on the concepts of fuzzy CI, interval-valued CI, and the definition of IT2FSs, which are constructed by infinite embedded type-1 fuzzy sets, a new method has been proposed by Havens et al. [27]. The other aggregation function, which is considered in this paper, proposed by Yager [28], is power average (PA) operator. Through this operator, the values are able to support each other or interact together and it is caused to the resulted value is more compliant with reality. It is extended to different kinds, including power weighted average operator (PWA), power ordered weighted average operator (POWA), power hybrid average operators (PHA), power geometric operator (PGA), power geometric weighted average (PGWA) and power ordered weighted geometric (POWG) operator [28, 29]. It has been, also generalized by Wan et al. [30, 31] to deal with intuitionistic fuzzy numbers. The rest of this paper is organized as follows:

In section 2, we will review some concepts which are necessary in other sections. Section 3 presents a new ranking method for IT2FSs. CI and PA operator will be used to aggregate a set of IT2FSs is Sections 4 and 5, respectively. Section 6 proposed a new way to determine the importance degree of decision makers or criteria. Sections 7 and 8 are dedicated to the application of these new concepts to MAGDM problems and a numerical example, respectively. The paper will be concluded in Section 9.

2 Some required definitions and preface

Let X be a reference set, a type-1 fuzzy subset of X namely $\tilde{A}$ is defined by the membership function (MF) $\mu_A : X \to [0,1]$ where, $\mu_A(x)$ $\forall x \in X$, expresses the membership degree of the element x in $\tilde{A}$, i.e.

$$\tilde{A} = \{(x, \mu_A(x)) \mid x \in X, \}.$$

The centroid point of type-1 fuzzy numbers (T1FNs) is an important property, which may be used to rank such numbers. Suppose $\tilde{A} = (a_1, a_2, a_3, a_4; w_a)$ be a T1TrFN. Its centroid point is determined as follows [32]:

$$\bar{x}_0 = \frac{1}{3}[a_1 + a_2 + a_3 + a_4 - \frac{a_3 a_4 - a_1 a_2}{(a_3 + a_4) - (a_1 + a_2)}], \quad \bar{y}_0 = \frac{w_a}{3}[1 + \frac{a_3 - a_2}{(a_3 + a_4) - (a_1 + a_2)}].$$

Then, $R(A) = \overline{x_0}.\overline{y_0}$ is called the ranking order of $\tilde{A}$.

Definition 2.1 [4] Let X be a universal set, $x \in X$ and, $u \in J_x \subset [0,1]$, then,

$$\tilde{\tilde{A}} = \{((x,u), \mu_{\tilde{\tilde{A}}}(x,u)) \mid \forall x \in X, \forall u \in J_x \subset [0,1]\},$$

is called a T2FS in which, $0 \le \mu_{\tilde{\tilde{A}}}(x,u) \le 1$ and it is a type-2 membership function.

From here onwards, for simplicity, we apply $\tilde{A}$ rather than $\tilde{\tilde{A}}$ to display a T2FS, and we use a particular case of T2FS, called IT2FSs, where $\forall(x,u)\ \mu_{\tilde{A}}(x,u)=1$, i.e.

$$\tilde{A}=\int_{x\in X}\int_{u\in J_x}1/(x,u)=\int_{x\in X}[\int_{u\in J_x}1/u]/x,$$

$J_x\subset[0,1]$ is primary membership of x and $\int_{u\in J_x}1/u$ is the secondary membership function (MF) at x [33].

As we see, in an IT2FS, the secondary membership function doesn't have any more information about uncertainty because the secondary grades for all $x\in X$ are equal 1. Then it is suitable to omit the 3rd dimension of IT2FSs (Fig. 1). For each IT2FS $\tilde{A}$ the union of all its primary memberships is called footprint of uncertainty of $\tilde{A}$ (FOU($\tilde{A}$)), i.e., $FOU(\tilde{A})=\bigcup_{x\in X}J_x$ (the shaded region in Fig. 1).

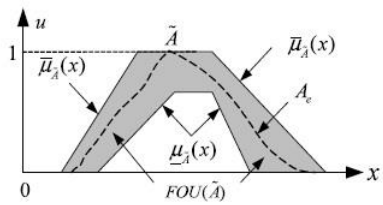

Fig. 1 Fuzzy sets

For each IT2FS $\tilde{A}$ its FOU is a bounded region. The upper bound, denoted by $\overline{\mu}_{\tilde{A}}(x)$ for all $x\in X$, is called the upper membership function (UMF) and the lower membership function (LMF) is the lower bound of FOU which is displayed by $\underline{\mu}_{\tilde{A}}(x)$, for all $x\in X$. UMF and LMF are both T1FSs and between them, there are infinite T1FSs that are called embedded T1FSs, displayed by A_e. When the UMF and LMF are trapezoidal /triangular the IT2FS is called trapezoidal/triangular, IT2FS (TrIT2FS/TIT2FS). Fig. 2 shows a TrIT2FS $\tilde{A}$, presented by [34]

$$\tilde{A}=(A^U,A^L)=((a_1^U,a_2^U,a_3^U,a_4^U;H_1(A^U),H_2(A^U)),\ \ (a_1^L,a_2^L,a_3^L,a_4^L;H_1(A^L),H_2(A^L)))$$

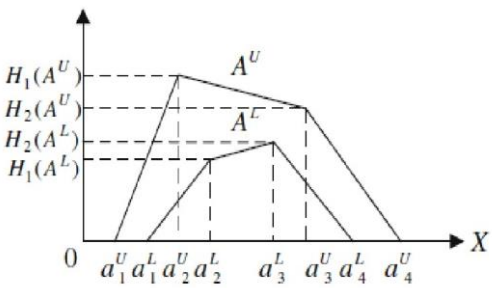

Fig. 2 TrIT2FS $\tilde{A}=\left(A^U,A^L\right)$

The arithmetic operations between TrIT2FSs have been defined as follows [17,18,34]:

Definition 2.2 Let

$$\tilde{B}_i = (B_i^U, B_i^L) = \langle (b_{i1}^U, b_{i2}^U, b_{i3}^U, b_{i4}^U; H_1(B_i^U), H_2(B_i^U)), (b_{i1}^L, b_{i2}^L, b_{i3}^L, b_{i4}^L; H_1(B_i^L), H_2(B_i^L)) \rangle \quad (i = 1, 2)$$

be two TrIT2FSs. Then

$$i)\ \tilde{B}_1 \oplus \tilde{B}_2 = (B_1^U, B_1^L) \oplus (B_2^U, B_2^L) = \langle (b_{11}^U + b_{21}^U, b_{12}^U + b_{22}^U, b_{13}^U + b_{23}^U, b_{14}^U + b_{24}^U; min(H_1(B_1^U), H_1(B_2^U)), min(H_2(B_1^U), H_2(B_2^U))), b_{11}^L + b_{21}^L, b_{12}^L + b_{22}^L, b_{13}^L + b_{23}^L, b_{14}^L + b_{24}^L; min(H_1(B_1^L), H_1(B_2^L)), min(H_2(B_1^L), H_2(B_2^L))) \rangle,$$

$$ii)\ \tilde{B}_1 \ominus \tilde{B}_2 = (B_1^U, B_1^L) \ominus (B_2^U, B_2^L) = \langle (b_{11}^U - b_{24}^U, b_{12}^U - b_{23}^U, b_{13}^U - b_{22}^U, b_{14}^U - b_{21}^U; min(H_1(B_1^U), H_1(B_2^U)), min(H_2(B_1^U), H_2(B_2^U))), (b_{11}^L - b_{24}^L, b_{12}^L - b_{23}^L, b_{13}^L - b_{22}^L, b_{14}^L - b_{21}^L; min(H_1(B_1^L), H_1(B_2^L)), min(H_2(B_1^L), H_2(B_2^L))) \rangle,$$

$$iii)\ \tilde{B}_1 \otimes \tilde{B}_2 = (B_1^U, B_1^L) \otimes (B_2^U, B_2^L) = \langle (b_{11}^U \times b_{21}^U, b_{12}^U \times b_{22}^U, b_{13}^U \times b_{23}^U, b_{14}^U \times b_{24}^U; min(H_1(B_1^U), H_1(B_2^U)), min(H_2(B_1^U), H_2(B_2^U))), (b_{11}^L \times b_{21}^L, b_{12}^L \times b_{22}^L, b_{13}^L \times b_{23}^L, b_{14}^L \times b_{24}^L; min(H_1(B_1^L), H_1(B_2^L)), min(H_2(B_1^L), H_2(B_2^L))) \rangle,$$

$$iv)\ k\tilde{B}_1 = \langle (k \times b_{11}^U, k \times b_{12}^U, k \times b_{13}^U, k \times b_{14}^U; H_1(B_1^U), H_2(B_1^U)), k \times (b_{11}^L, k \times b_{12}^L, k \times b_{13}^L, k \times b_{14}^L; H_1(B_1^L), H_2(B_1^L)) \rangle, \text{where} k>0.$$

In many real applications of IT2FSs, it is necessary to compare them. The first method for ranking of T2FSs was introduced by Mitchell [35], then, Wu and Mendel [36] offered a centroid based ranking method. Also, we can see some other methods to rank TrIT2FSs in [17, 34, 19].

Let $\tilde{A} = (A^U, A^L) = \langle (a_1^U, a_2^U, a_3^U, a_4^U; H_1(A^U), H_2(A^U)), (a_1^L, a_2^L, a_3^L, a_4^L; H_1(A^L), H_2(A^L)) \rangle$ be a TrIT2FS its ranking value represented by Rank$(\tilde{A})$, is defined as [37]:

$$Rank(\tilde{A}) = M_1(A^U) + M_1(A^L) + M_2(A^U) + M_2(A^L) + M_3(A^U) + M_3(A^L) - \frac{1}{4}(S_1(A^U) + S_1(A^L) + S_2(A^U) + S_2(A^L) + S_3(A^U) + S_3(A^L) + S_4(A^U) + S_4(A^L)) + H_1(A^U) + H_2(A^U) + H_1(A^L) + H_2(A^L),$$

where, for $j \in \{U, L\}$

$$M_p(A^j) = (a_p^j + a_{p+1}^j)/2, 1 \le p \le 3, \quad S_q(A^j) = \sqrt{\frac{1}{2}\sum_{k=q}^{q+1}(a_k^j - \frac{1}{2}\sum_{k=q}^{q+1} a_k^j)^2}, 1 \le q \le 3$$

and

$$S_4(A^j) = \sqrt{\frac{1}{2}\sum_{k=1}^{4}(a_k^j - \frac{1}{2}\sum_{k=1}^{4}a_k^j)^2}.$$

Centroid, cardinality, fuzziness, variance, and skewness are five uncertainty measures for IT2FSs, which have been defined in [33]. Li et al. [38] considered a symmetrical TrIT2FS $\tilde{A}$ as in fig. 3, a general case with six parameters (a,b,c,d,m,h). Then, its uncertainty degree $\rho_{\tilde{A}}$ is

$$\rho_{\tilde{A}} = 1 - \frac{2b(a-c)-(a+c)(b-d)h}{(b-d)^2} + 2\frac{ab(b-d)h - b^2(a-c)}{h(b-d)^3} ln(1-h\frac{b-d}{b})$$

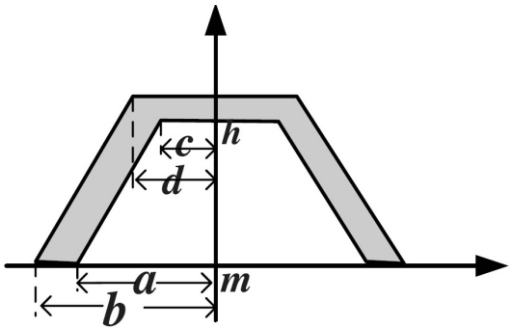

Fig. 3 Symmetrical TrIT2FS

To solve decision-making problems, each object must be evaluated from different aspects, called criteria or attribute, and then aggregate them to an amount. One of the most useful aggregation functions is CI, which enables scientists to consider the weight/importance of each group of criteria in aggregation process [26, 39]. Obtaining measure of each non-empty set of attributes, while we know each one's weight, has been explained by Tzeng and Huang [26].

Definition 2.3 [39] Suppose $X = \{x_1, x_2, ..., x_n\}$ be the reference set with the power set $P(X)$, f be a function on X with values $f_1, f_2, ..., f_n$, and μ be a monotone measure on $P(X)$, then

$$(C)\int f d\mu = \sum_{i=1}^{n}[f_{i^*} - f_{(i-1)^*}].\mu(\{x_{i^*}, x_{(i+1)^*}, .., x_{n^*}\}),$$

where, $f_{0^*} = 0$ and $\{1^*, 2^*, ..., n^*\}$ is a permutation of $\{1, 2, ..., n\}$ such that

$$f_{1^*} \le f_{2^*} \le ... \le f_{n^*}.$$

Also, Yager [28] introduced power average (PA) operator, as follows:

Definition 2.4 Let $u_1, u_2, \ldots, u_n$ be a collection of values to be aggregated, then

$$PA(u_1, u_2, \ldots, u_n) = \frac{\sum_{i=1}^{n}(1+T(u_i))u_i}{\sum_{i=1}^{n}(1+T(u_i))},$$

where,

$$T(u_i) = \sum_{\substack{j=1 \\ j \neq i}}^{n} Supp(u_i, u_j),$$

and $Supp(u,v)$ is called support function, denoted the support of values u and v of each other and satisfied in the following three properties:

1) $Supp(u,v) = Supp(v,u)$;

2) $Supp(u,v) \in [0,1]$;

3) If $|x-y| < |u-v|$, then $Supp(u,v) < Supp(x,y)$;

3 A new ranking method for TrIT2FSs

In this section, we extend the proposed ranking method for T1FNs by Wang et al. [32]. Then, we update the expressed formula for computing uncertainty degree of general TrIT2FSs. Finally, a new method which is constructed from previous sense, is introduced to rank TrIT2FSs.

Definition 3.1 Suppose $\tilde{U} = (u_1, u_2, u_3, u_4; w_u)$ and $\tilde{V} = (v_1, v_2, v_3, v_4; w_v)$ be T1TrFNs in which (x_{0u}, y_{0u}) and (x_{0v}, y_{0v}) are their centroid points respectively, then:

i) If $x_{0u} > x_{0v}$ ($x_{0u} < x_{0v}$) then $\tilde{U} \succ \tilde{V}$ ($\tilde{U} \prec \tilde{V}$),

ii) If $x_{0u} = x_{0v}$ and $y_{0u} = y_{0v}$, then $\tilde{U} \approx \tilde{V}$,

iii) If $x_{0u} = x_{0v}$ then, $\tilde{U} \succ \tilde{V}$ ($\tilde{U} \prec \tilde{V}$) is resulted from $y_{0u} > y_{0v}$ ($y_{0u} < y_{0v}$).

Definition 3.2 (i**)** Suppose $\tilde{U} = (u_1, u_2, u_3, u_4; h_{1u}, h_{2u})$ and $\tilde{V} = (v_1, v_2, v_3, v_4; h_{1v}, h_{2v})$ be T1TrFNs, Euclidean distance (d_e) and Hamming distance (d_h) between them have been defined as follows:

$$d_e(\tilde{U}, \tilde{V}) = \sqrt{\frac{1}{4}\left[(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 + (u_4 - v_4)^2 + \max\left\{|h_{1u} - h_{1v}|, |h_{2u} - h_{2v}|\right\}\right]},$$

$$d_h(\tilde{U}, \tilde{V}) = \frac{1}{4}\left[|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3| + |u_4 - v_4| + \max\left\{|h_{1u} - h_{1v}|, |h_{2u} - h_{2v}|\right\}\right];$$

(ii**)** Let $\tilde{A} = (A^U, A^L)$ and $\tilde{B} = (B^U, B^L)$ be TrIT2FNs. Euclidean distance (d_e) and Hamming distance (d_h) between them have been defined as follows:

$$d_h\left(\tilde{A}, \tilde{B}\right) = \lambda d_h\left(A^U, B^U\right) + (1-\lambda) d_h\left(A^L, B^L\right)$$

$$d_e\left(\tilde{A}, \tilde{B}\right) = \lambda d_e\left(A^U, B^U\right) + (1-\lambda) d_e\left(A^L, B^L\right)$$

where, $\lambda \in (0,1)$ and determined by decision maker.

Uncertainty degree's formula that is introduced by Li et al. [38], is specialized only for symmetric TrIT2FSs. Based on the definition of uncertainty measure, we can get a similar formula for generalized TrIT2FSs. For each $\alpha \in [0,1]$, the related uncertainty degree i.e., $\rho_{\tilde{A}}(\alpha)$, is defined as $\rho_{\tilde{A}}(\alpha) = 1 - \frac{|\tilde{A}_\alpha^L|}{|\tilde{A}_\alpha^U|}$, where,

$\tilde{A} = \langle (a_1^U, a_2^U, a_3^U, a_4^U; 1,1), (a_1^L, a_2^L, a_3^L, a_4^L; h, h) \rangle$ and $|.|$ is a measure such as Lebuge measure (fig. 5).

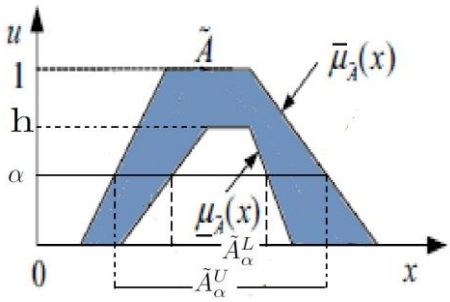

Fig. 4 α -cut of TrIT2FS $\tilde{A}$

Then, using

$$\rho_{\tilde{A}} = \int_0^1 2\alpha \rho_{\tilde{A}}(\alpha) d\alpha = 1 - \int_0^h 2\alpha \frac{a_4^L - a_1^L - (a_4^L - a_3^L + a_2^L - a_1^L)\frac{\alpha}{h}}{a_4^U - a_1^U - (a_4^U - a_3^U + a_2^U - a_1^U)\alpha} d\alpha,$$

we can obtain a generalized formula to compute uncertainty degree of each TrIT2FS:

$$\rho_{\tilde{A}} = 1 - \frac{2n}{qh}[\frac{1}{2}h^2 + (\frac{p}{q} - \frac{mh}{n})h + \frac{p}{q}(\frac{p}{q} - \frac{mh}{n}) ln|1 - \frac{q}{p}h|], \quad (1)$$

where, $m = a_4^L - a_1^L, n = a_4^L - a_3^L + a_2^L - a_1^L, p = a_4^U - a_1^U, q = a_4^U - a_3^U + a_2^U - a_1^U$.

Now, we are ready to propose an algorithm for ranking TrIT2FSs.

A new ranking algorithm

As we know, a TrIT2FS is displayed by its FOU which is bounded by UMF and LMF. These bounds are trapezoidal T1FNs. Let $\tilde{A} = \langle \bar{\mu}_{\tilde{A}}, \underline{\mu}_{\tilde{A}} \rangle$ and $\tilde{B} = \langle \bar{\mu}_{\tilde{B}}, \underline{\mu}_{\tilde{B}} \rangle$ be two IT2FSs. Suppose that $\bar{\mu}_{\tilde{A}}$ and $\bar{\mu}_{\tilde{B}}$ be the upper membership functions of their FOU s, respectively. Likewise, their lower membership functions are displayed by $\underline{\mu}_{\tilde{A}}$ and $\underline{\mu}_{\tilde{B}}$. $\tilde{A}$ and $\tilde{B}$ are compared through the following steps:

Step one: Pick $\bar{\mu}_{\tilde{A}}$ and $\bar{\mu}_{\tilde{B}}$ which are T1FSs and compare them using the proposed methods for T1FSs (Def. 3.1). The greater one shows that the corresponding IT2FS is larger and if they are equal go to step two.

Step two: Get the lower membership functions of IT2FSs and compare them using Def. 3.1. Ranking of given IT2FSs are like to their LMF's ranking if they aren't equal, and then go to Step three.

Step three: Compute the uncertainty degrees of given IT2FSs using Eq. 1. The greater one has a small uncertainty degree, else, the IT2FSs $\tilde{A}$ and $\tilde{B}$ are equal.

The above algorithm has some properties, which may be proved as follows.

Theorem 1. For TrIT2FSs $\tilde{A}, \tilde{B}$ and $\tilde{C}$,

1) If $\tilde{A} \succeq \tilde{B}$ and $\tilde{A} \preceq \tilde{B}$, then $\tilde{A} \approx \tilde{B}$;
2) If $\tilde{A} \succeq \tilde{B}$ and $\tilde{B} \succeq \tilde{C}$, then $\tilde{A} \succeq \tilde{C}$;

3) If $\tilde{A} \succeq \tilde{B}$, then $\tilde{A}+\tilde{C} \succeq \tilde{B}+\tilde{C}$;

where, $\succeq$ means larger than or equal in the sense of ranking and $\approx$ means the same ranking.

Proof Proving 1) and 2) are easy, we prove only 3).

Let $\tilde{A} \succeq \tilde{B}$. If this ranking order is obtained from the ranking order of their UMFs or LMFs, then from ranking order's properties of T1FSs, we derive that $\tilde{A}+\tilde{C} \succeq \tilde{B}+\tilde{C}$. If the ranking order of their UMFs and LMFs are equal and $\rho_{\tilde{A}} < \rho_{\tilde{B}}$. It is easy to see that

$$\rho_{\tilde{A}+\tilde{C}} = 1-\frac{2n_{\tilde{A}+\tilde{C}}}{q_{\tilde{A}+\tilde{C}}h}[\frac{1}{2}h^2+(\frac{p_{\tilde{A}+\tilde{C}}}{q_{\tilde{A}+\tilde{C}}}-\frac{m_{\tilde{A}+\tilde{C}}h}{n_{\tilde{A}+\tilde{C}}})h+\frac{p_{\tilde{A}+\tilde{C}}}{q_{\tilde{A}+\tilde{C}}}(\frac{p_{\tilde{A}+\tilde{C}}}{q_{\tilde{A}+\tilde{C}}}-\frac{m_{\tilde{A}+\tilde{C}}h}{n_{\tilde{A}+\tilde{C}}})ln|1-\frac{q_{\tilde{A}+\tilde{C}}}{p_{\tilde{A}+\tilde{C}}}h|],$$

where,

$$h = min\{h_{\tilde{A}}, h_{\tilde{C}}\},\ \ m_{\tilde{A}+\tilde{C}} = a_4^L - a_1^L + c_4^L - c_1^L,\ \ n_{\tilde{A}+\tilde{C}} = a_4^L - a_3^L + a_2^L - a_1^L + c_4^L - c_3^L + c_2^L - c_1^L,$$
$$p_{\tilde{A}+\tilde{C}} = a_4^U - a_1^U + c_4^U - c_1^U,\ \ q_{\tilde{A}+\tilde{C}} = a_4^U - a_3^U + a_2^U - a_1^U + c_4^U - c_3^U + c_2^U - c_1^U.$$

Similarly,

$$\rho_{\tilde{B}+\tilde{C}} = 1-\frac{2n_{\tilde{B}+\tilde{C}}}{q_{\tilde{B}+\tilde{C}}h}[\frac{1}{2}h^2+(\frac{p_{\tilde{B}+\tilde{C}}}{q_{\tilde{B}+\tilde{C}}}-\frac{m_{\tilde{B}+\tilde{C}}h}{n_{\tilde{B}+\tilde{C}}})h+\frac{p_{\tilde{B}+\tilde{C}}}{q_{\tilde{B}+\tilde{C}}}(\frac{p_{\tilde{B}+\tilde{C}}}{q_{\tilde{B}+\tilde{C}}}-\frac{m_{\tilde{B}+\tilde{C}}h}{n_{\tilde{B}+\tilde{C}}})ln|1-\frac{q_{\tilde{B}+\tilde{C}}}{p_{\tilde{B}+\tilde{C}}}h|],$$

where,

$$h = min\{h_{\tilde{B}}, h_{\tilde{C}}\},\ \ m_{\tilde{B}+\tilde{C}} = a_4^L - a_1^L + c_4^L - c_1^L,\ \ n_{\tilde{B}+\tilde{C}} = a_4^L - a_3^L + a_2^L - a_1^L + c_4^L - c_3^L + c_2^L - c_1^L,$$
$$p_{\tilde{B}+\tilde{C}} = a_4^U - a_1^U + c_4^U - c_1^U,\ \ q_{\tilde{B}+\tilde{C}} = a_4^U - a_3^U + a_2^U - a_1^U + c_4^U - c_3^U + c_2^U - c_1^U.$$

As we can see, the explained parameters to compute uncertainty degrees of $\tilde{A}+\tilde{C}$ and $\tilde{B}+\tilde{C}$ are the resulting of the same shift in their related parameters of $\tilde{A}$ and $\tilde{B}$. Then, from $\rho_{\tilde{A}} < \rho_{\tilde{B}}$, we conclude that $\rho_{\tilde{A}+\tilde{C}} < \rho_{\tilde{B}+\tilde{C}}$.

The following subdivision shows contrast examples between Chen 2010 [3], Chen 2012 [1] and the proposed method.

3.1 Comparison examples

We compare the proposed method with other existing ranking methods through the following examples.

Example 3.1 Let $\tilde{A} = \langle(1,5,6,10;1),(4,5,6,7;1)\rangle$ and $\tilde{B} = \langle(2,5,6,9;1),(3,5,6,8;1)\rangle$ be TrIT2FSs (Fig. 6).

Based on Chen's method [34], $\tilde{A} < \tilde{B}$, the proposed ranking method in [17] resulted that $\tilde{A} = \tilde{B}$. Comparing them by the proposed algorithm in this article, gives $\tilde{A} < \tilde{B}$ because $\rho_{\tilde{B}} < \rho_{\tilde{A}}$.

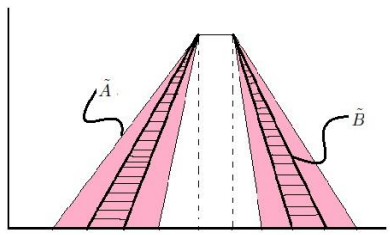

Fig. 5 TrIT2FS $\tilde{A}$ and $\tilde{B}$

Example 3.2 Suppose $\tilde{A} = \langle (1,4,5.75;1),(3,4.25,5;.8) \rangle$ and $\tilde{B} = \langle (1,4,5.75;1),(2,3.6,5;.7) \rangle$.

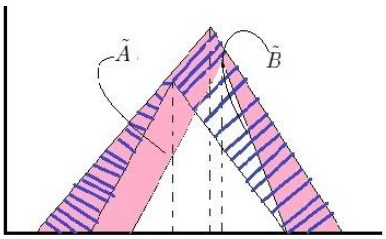

Fig. 6 TIT2FS $\tilde{A}$ and $\tilde{B}$

As we can see (Fig. 7), their UMFs are equal based on the ranking order of their LMFs, we obtain $\tilde{A} > \tilde{B}$. It is similar to what determined using the proposed methods in [17, 34].

Example 3.3 Suppose $\tilde{A} = \langle (0,4,8;1),(2,4,6;.9) \rangle$ and $\tilde{B} = \langle (1,4,7;1),(3,4,5;.9) \rangle$. According to our proposed method, their centroid points are the same, but, $\tilde{A}$ has a larger uncertainty degree than $\tilde{B}$, then $\tilde{A} < \tilde{B}$. This ranking order is like to what obtained from [17], and is opposite with [22]. We believe $\tilde{A} < \tilde{B}$ is more reasonable than $\tilde{A} > \tilde{B}$, because $\tilde{A}$ is only, more uncertain than $\tilde{B}$.

4 Aggregation IT2FSs using CI

Application of CI in aggregation process of IT2FSs will be argued in this section.

Definition 4.1 Consider $X = \{x_1, x_2, ..., x_n\}$ with power set $P(X)$ be the reference set, $\tilde{f}$ be an IT2FS-valued function on X as $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_n$, and define a monotone measure μ on $P(X)$ then,

$$(C)\int f d\mu = \sum_{i=1}^{n} [\tilde{f}_{i^*} - \tilde{f}_{(i-1)^*}].\mu(\{x_{i^*}, x_{(i+1)^*}, .., x_{n^*}\}),$$

where, $\tilde{f}_{0^*} = \langle (0,0,0,0;1),(0,0,0,0;1) \rangle, \mu(\{x_{i^*}, x_{(i+1)^*}, .., x_{n^*}\})$ is joint measure of $\{x_{i^*}, x_{(i+1)^*}, .., x_{n^*}\}$ and $\{x_{1^*}, x_{2^*}, ..., x_{n^*}\}$ is a permutation of $\{x_1, x_2, ..., x_n\}$ such that

$$\tilde{f}_{1^*} \le \tilde{f}_{2^*} \le ... \le \tilde{f}_{n^*}.$$

The output of CI in Def. 4.1 is an IT2FS and then, its FOU, which is denoted by FOU(CI $\tilde{f}$), is a bounded region. Due to the lexicographic ranking method, its UMF is the CI of UMFs of $\tilde{f}_i, i = 1,2,\cdots n$, but its LMF isn't necessary, the Choquet integral of LMFs. Indeed, they are created from UMFs and LMFs of $\tilde{f}(x_i), i = 1,2,\cdots n$, respectively, using the Def. 4.1. This is because, there may be some IT2FSs such as $\tilde{f}_{i^*}$ and $\tilde{f}_{(i+1)^*}$ with conditions $UMF(\tilde{f}_{i^*}) < UMF(\tilde{f}_{(i+1)^*})$ and $LMF(\tilde{f}_{i^*}) > LMF(\tilde{f}_{(i+1)^*})$. Then, $\tilde{f}_{i^*} < \tilde{f}_{(i+1)^*}$ and this ranking order is used to compute CI of UMFs of $\tilde{f}_i, (i = 1,2,\cdots n)$, independently, but to obtain the CI of LMFs of $\tilde{f}_i, (i = 1,2,\cdots n)$, separately, the proposed ranking order for IT2FSs isn't

applicable. If $\tilde{f}_i, (i=1,2,\cdots n)$, are ranked only based on their uncertainty degrees, then the UMF and LMF of FOU(CI $\tilde{f}$) are the CI of UMFs and LMFs of $\tilde{f}_i, (i=1,2,\cdots n)$, respectively.

It should be noted that with respect to the above Definition, we can use the computed values from CI in other processes, easily. Also, CI of IT2FNs has properties that are mentioned in Section two for CI in general. In the following theorem, we will consider only monotonicity property of CI, the other one can be obtained easily.

Theorem 2. Let $\tilde{f}$ and $\tilde{g}$ be two IT2FSs-valued functions on $X=\{x_1,x_2,...,x_n\}$, with $\tilde{f}_1 \le \tilde{g}_1, \tilde{f}_2 \le \tilde{g}_2,..., \tilde{f}_n \le \tilde{g}_n$ i.e. $\tilde{f} \le \tilde{g}$. Then, for each monotone measure μ on $P(X)$, we have:

$$(C)\int \tilde{f} d\mu \le (C)\int \tilde{g} d\mu.$$

Proof. Suppose $\mu\{x_i\}$ for $i=1,2,\cdots,n$ are given. Then, we can compute the measure of each element of $P(X)$. Without loss of generality, let $\tilde{f}_1 \le \tilde{f}_2 \le ... \le \tilde{f}_n$, we have

$$(C)\int \tilde{f} d\mu = \tilde{f}_1 \times [\mu\{X\} - \mu\{x_2,x_3,\cdots,x_n\}] + \tilde{f}_2 \times [\mu\{x_2,x_3,\cdots,x_n\} - \mu\{x_3,\cdots,x_n\}] + \cdots + \tilde{f}_n \times [\mu\{x_n\} - \mu\{x_{n+1}\}],$$

where, $\mu\{x_{n+1}\}=0.$ In the other hand, we know that $\tilde{f}_1 \le \tilde{g}_1, \tilde{f}_2 \le \tilde{g}_2,..., \tilde{f}_n \le \tilde{g}_n$. Therefore,

$$\tilde{f}_1 \times [\mu\{X\} - \mu\{x_2,x_3,\cdots,x_n\}] + \tilde{f}_2 \times [\mu\{x_2,x_3,\cdots,x_n\} - \mu\{x_3,\cdots,x_n\}] + \cdots + \tilde{f}_n \times [\mu\{x_n\} - \mu\{x_{n+1}\}] \le \tilde{g}_1 \times [\mu\{X\} - \mu\{x_2,x_3,\cdots,x_n\}] + \tilde{g}_2 \times [\mu\{x_2,x_3,\cdots,x_n\} - \mu\{x_3,\cdots,x_n\}] + \cdots + \tilde{g}_n \times [\mu\{x_n\} - \mu\{x_{n+1}\}].$$

Thus, we obtain

$$(C)\int \tilde{f} d\mu \le (C)\int \tilde{g} d\mu.$$

5 PA operator of IT2FNs

Aggregation process of IT2FNs using PA operator will be argued in this section.

5.1 A new method to aggregate IT2FNs based on PA operator

Definition 5.1 Let $\tilde{A}_i, i=1,2,\cdots,n\,(n \ge 3)$ be IT2FNs, the PA of them is defined as follows:

$$PAIT2FN(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n) = \frac{\sum_{i=1}^{n}[(1+T(\tilde{A}_i))\tilde{A}_i]}{\sum_{i=1}^{n}(1+T(\tilde{A}_i))},$$

where, $T(\tilde{A}_i) = \sum_{\substack{j=1 \\ j\ne i}}^{n} Supp(\tilde{A}_i,\tilde{A}_j)$ and for each i, j, $Supp(\tilde{A}_i,\tilde{A}_j)$ is the support of two IT2FN, satisfying in the following properties:

1) $Supp(\tilde{A}_i,\tilde{A}_j) = Supp(\tilde{A}_j,\tilde{A}_i)$;

2) $Supp(\tilde{A}_i, \tilde{A}_j) \in [0,1]$;

3) If $d_h(\tilde{A}_i, \tilde{A}_j) < d_h(\tilde{A}_p, \tilde{A}_q)$ then $Supp(\tilde{A}_i, \tilde{A}_j) > Supp(\tilde{A}_p, \tilde{A}_q)$, where d_h is the Hamming distance as in Def. 3.2 that can be replaced by d_e (Euclidean distance).

In the following theorem, we'll show that the $PAIT2FN(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ has the mentioned properties of PA operator:

Theorem 3. Suppose $\tilde{A}_i, i = 1, 2, \cdots, n\ (n \geq 3)$ to be INT2FNs which their power average is denoted by $PAIT2FN(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ then,

1) PA operator is bounded, i.e.

$$min(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) \leq PAIT2FN(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) \leq max(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n);$$

2) If for each $i \neq j$ we had $Supp(\tilde{A}_i, \tilde{A}_j) = k$ then, PA converted to arithmetic average, i.e.

$$PAIT2FN(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) = \frac{\sum_{i=1}^{n} \tilde{A}_i}{n};$$

3) PA is idempotent i.e.

$$PAIT2FN(\tilde{A}, \tilde{A}, \ldots, \tilde{A}) = \tilde{A};$$

4) Let $\{\tilde{A}_{1*}, \tilde{A}_{2*}, \ldots, \tilde{A}_{n*}\}$ be a permutation of $\{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\}$, then

$$PAIT2FN(\tilde{A}_{1*}, \tilde{A}_{2*}, \ldots, \tilde{A}_{n*}) = PAIT2FN(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n).$$

Proof.

1) Let $\tilde{A}_{1*} = min(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n), \tilde{A}_{n*} = max(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ and $w_i = \dfrac{1 + T(\tilde{A}_i)}{\sum_{i=1}^{n}(1 + T(\tilde{A}_i))}$. It is easy to see that $PAIT2FN(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) = \sum_{i=1}^{n} w_i \tilde{A}_i$, $\sum_{i=1}^{n} w_i = 1$ and $w_i \geq 0, (i = 1, 2, \cdots, n)$. In the other hand $\tilde{A}_{1*} \leq \tilde{A}_{i*} \leq \tilde{A}_{n*}$ for $(i = 1, 2, \cdots, n)$, then

$$\tilde{A}_{1*} = \sum_{i=1}^{n} w_i \tilde{A}_{1*} \leq \sum_{i=1}^{n} w_i \tilde{A}_i = PAIT2FN(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) \leq \sum_{i=1}^{n} w_i \tilde{A}_{n*} = \tilde{A}_{n*}.$$

2) Let $Supp(\tilde{A}_i, \tilde{A}_j) = k$ for each $i \neq j$ is satisfied. Then based on the Def. 4.1, we have $T(\tilde{A}_i) = (n-1)k$ and

$$PAIT2FN(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) = \frac{\sum_{i=1}^{n}(1 + (n-1)k)\tilde{A}_i}{\sum_{i=1}^{n}(1 + (n-1)k)} = \frac{\sum_{i=1}^{n}(1 + (n-1)k)\tilde{A}_i}{n(1 + (n-1)k)} = \frac{\sum_{i=1}^{n}\tilde{A}_i}{n}.$$

3) Suppose $\tilde{A}_i = \tilde{A}, i = 1, 2, \cdots, n\ (n \geq 3)$ to be n IT2FNs which are the same. Then for all i and j we have $Supp(\tilde{A}_i, \tilde{A}_j) = k$. Finally, as in part (2)

$$PAIT2FN(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n)=\frac{\sum_{i=1}^{n}\tilde{A}_i}{n}=\frac{\sum_{i=1}^{n}\tilde{A}}{n}=\tilde{A}.$$

4) Based on Def. 5.1, it is obvious.

If the numbers have different importance, PA is extended to weighted PA which can be defined as follows:

Definition 5.2 Consider IT2FNs $\tilde{A}_i, i=1,2,\cdots,n\,(n\geq 3)$ with importance vector $W=(w_1,w_2,\cdots,w_n)$ such that $\sum_{i=1}^{n}w_i=1$, then

$$PAIT2FN_w(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n)=\frac{\sum_{i=1}^{n}[(1+w_iT(\tilde{A}_i))w_i\tilde{A}_i]}{\sum_{i=1}^{n}(1+w_iT(\tilde{A}_i))w_i},$$

where, $T(\tilde{A}_i)=\sum_{\substack{j=1\\ j\neq i}}^{n}Supp(\tilde{A}_i,\tilde{A}_j)$.

Also, power order weighted average operator of IT2FNs based on the OWA operator [28] is defined as:

Definition 5.3 Consider $\tilde{A}_i, i=1,2,\cdots,n\,(n\geq 3)$ with a permutation $\tilde{A}_{i^*}, i=1,2,\cdots,n$ such that $\tilde{A}_{1^*}\leq\tilde{A}_{2^*}\leq\cdots\leq\tilde{A}_{n^*}$ be IT2FNs, then

$$POWAIT2FN(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n)=\sum_{k=1}^{n}\omega_k\tilde{A}_{k^*},$$

where

$$\omega_k=Q(R_k/TV)-Q(R_{k-1}/TV),R_k=\sum_{j=1}^{k}V(\tilde{A}_{j^*}),TV=\sum_{j=1}^{n}V(\tilde{A}_{j^*}),V(\tilde{A}_{j^*})=1+T(\tilde{A}_{j^*}),$$

$Q:[0,1]\rightarrow[0,1]$ is a basic unit interval monotonic (BUM) function with following properties: $Q(0)=0,Q(1)=1$ and for each $x>y$ then $Q(x)>Q(y)$. The Support of j th largest IT2FN by all the other ones denoted by $T(\tilde{A}_{j^*})$ i.e., $T(\tilde{A}_{j^*})=\sum_{\substack{l=1\\ l\neq j}}^{n}Supp(\tilde{A}_{l^*},\tilde{A}_{j^*})$, and the support of l th largest value for j th largest value indicated by $Supp(\tilde{A}_{l^*},\tilde{A}_{j^*})$.

Note: In the above definition:

1) If $Q(x)=x$ then $\omega_k=V(\tilde{A}_{k^*})/TV=[1+T(\tilde{A}_{k^*})]/\sum_{j=1}^{n}[1+T(\tilde{A}_{j^*})]$ and thus

$$POWAIT2FN(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n)=PAIT2FN(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n).$$

2) If $Q(x)=x$ and $Supp(\tilde{A}_{l^*},\tilde{A}_{j^*})=c,(c\in[0,1],l\neq j)$, then

$$\omega_k = V(\tilde{A}_{k^*})/TV = [1+T(\tilde{A}_{k^*})]/\sum_{j=1}^{n}[1+T(\tilde{A}_{j^*})] = \frac{1}{m},$$

and then POWAIT2FN operator reduces to arithmetic average operator of IT2FNs i.e.

$$POWAIT2FN(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) = \sum_{i=1}^{n}\tilde{A}_i / m.$$

3) For $Q(x)=1$ and $Q(x)=0$ the POWAIT2FN operator reduces to *max* and *min* operators, respectively. Also, it is easy to see that POWAIT2FN operator has properties such as boundary, commutatively and idempotency.

It is possible that the given arguments have different importance, and then, the POWAIT2FN operator is extended with a hybrid operator called the power hybrid average operator of IT2FNs and defined as follows:

Definition 5.4 The power hybrid average operator of IT2FNs (PHAIT2FN) $\tilde{A}_i, i=1,2,\cdots,n\,(n\geq 3)$ is defined as

$$PHAIT2FN_{\omega,w}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) = \sum_{k=1}^{n}\omega_k \tilde{A}'_{k^*},$$

where, $w=(w_1, w_2, \cdots, w_n)$ is the weight vector of $\tilde{A}_i (i=1,2,\cdots,n)$, with $0\leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ with $0\leq \omega_i \leq 1$ and $\sum_{i=1}^{n}\omega_i = 1$ is associated vector and obtain as in Def. 4.3, m in $\tilde{A}'_i = m w_i \tilde{A}_i (i=1,2,\cdots,n)$ called balancing coefficient and $\tilde{A}'_{k^*}$ is the k th largest of the weighted IT2FN $\tilde{A}'_k (i=1,2,\cdots,n)$.

The support of $\tilde{A}_i$ by all other IT2FNs is computed by $T(\tilde{A}_i)$ as follows:

An algorithm for obtaining $T(\tilde{A}_i)$

Suppose $\tilde{A}_i, i=1,2,\cdots,n\,(n\geq 3)$ be INT2FNs with a weight vector $W=(w_1, w_2, \cdots, w_n)$ then

Step one: For each pair of $\tilde{A}_i$ and $\tilde{A}_j, (i\neq j)$, compute the Hamming (Euclidean) distance between them as in Def. 3.1 which, are denoted by $d_{ij} = d_h(\tilde{A}_i, \tilde{A}_j)$ ($d_{ij} = d_e(\tilde{A}_i, \tilde{A}_j)$).

Step two: Compute relative distance $rd_{ij} = \dfrac{d_{ij}}{\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}}$.

Step three: Support for $\tilde{A}_i$ from $\tilde{A}_j$ is denoted by S_{ij} and obtain as $S_{ij} = Supp(\tilde{A}_i, \tilde{A}_i) = 1 - rd_{ij}$.

Step four: Calculate AS_i, called the average support of $\tilde{A}_i$ by all the other IT2FNs, where

$$AS_i = \frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} w_j S_{ij}.$$

Step five: Normalize the average support as follows:

$$T(\tilde{A}_i) = \frac{AS_i}{\sum_{j=1}^{n} AS_j}.$$

5.2 Comparative example

In this example, the proposed method compared with Chen's method [17, 34], where, arithmetic averaging or weighted arithmetic averaging methods has been used to obtain a single value from more than values. It means, a small (large) value can influence and decrees large (increase small) and toward values.

Let $\tilde{a} = \langle (4,5,6,7;1),(4.5,5,6,6.5;.8)\rangle, \tilde{b} = \langle (1,5,6,9;1),(3,4,5,6;.6)\rangle$ and $\tilde{c} = \langle (2,4,5,7;1),(3,4,5,6;.7)\rangle$ be three IT2FNs with weight values $w_a = 0.45, w_b = 0.25$ and $w_c = 0.3$. Based on Chen's method, we have

$$\tilde{x} = 0.45\langle (4,5,6,7;1),(4.5,5,6,6.5;.8)\rangle + 0.25\langle (1,5,6,9;1),(3,4,5,6;.6)\rangle + 0.3\langle (2,4,5,7;1),(3,4,5,6;.7)\rangle,$$

and then

$$\tilde{x} = \langle (2.6,4.7,5.7,7.5;1),(3.2,4.45,5.45,6.2;0.6)\rangle.$$

Based on our proposed method

$$PA(\tilde{a},\tilde{b},\tilde{c}) = \langle (2.3,4.7,5.7,7.7;1),(3.5,4.3,5.4,6.2;0.6)\rangle.$$

The obtained values from these two methods are almost equal, but there exist slight differences due attention to the distance between the input values.

6 A new method obtaining weight vector

CI is used to specify the weight of each DM in group decision-making problems.

Let $WC = (wc_1, wc_2, \cdots, wc_n)$ be the weight vector of criteria which is obtained through the direct question from the manager. Then, as we know each DM has its own expertise. So, we asked the manager to evaluate them up to all criteria and arrange these values in a matrix which is called weight matrix (WM). It means that we have:

$$WM = [wm_{ij}]_{k\times n} = \begin{array}{c} \\ D_1 \\ D_2 \\ \vdots \\ D_k \end{array} \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & \cdots & c_n \end{array} \\ \begin{pmatrix} wm_{11} & wm_{12} & \cdots & wm_{1n} \\ wm_{21} & wm_{22} & \cdots & wm_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ wm_{k1} & wm_{k2} & \cdots & wm_{kn} \end{pmatrix} \end{array}$$

where, k is the number of contributed DMs, n is the number of criteria and $wm_{ij} \in [0,1]$ is the crisp assessment value of i th DM against to j th criterion. Finally, using CI each row of this matrix is aggregated to a single value that is called the weight of the respected DM i.e.

$$wd_i = (C)\int wm_{i()} d\mu = \sum_{j=1}^{n} [wm_{ij}^* - wm_{ij-1}^*]\mu\{c_j^*, c_{j+1}^*, \cdots, c_n^*\},$$

where, $wm_{i0}=0$, $\{c_1^*,c_2^*,\cdots,c_n^*\}$ is the permutation of $\{c_1,c_2,\cdots,c_n\}$ such that $wm_{i1}^*\le wm_{i2}^*\le\cdots\le wm_{in}^*$, $\mu\{c_j^*,c_{j+1}^*,\cdots,c_n^*\}$ is called the measure of criteria $c_j^*,c_{j+1}^*,\cdots,c_n^*$ and calculated by λ-fuzzy measure, wd_i is the weight of ith DM and $WD=(wd_1,wd_2,\cdots,wd_k)$ is the weight vector of DMs.

On the other hand, each DM has their own skilled opinion corresponding to each criterion and we have to collect these values. It will be done as follows:
First, we construct an assessment matrix (AM):

$$AM=[am_{ij}]_{n\times k}=\begin{array}{c} \\ c_1\\ c_2\\ \vdots\\ c_n\end{array}\begin{array}{c} \begin{array}{cccc} D_1 & D_2 & \cdots & D_k\end{array}\\ \begin{pmatrix} am_{11} & am_{12} & \cdots & am_{1k}\\ am_{21} & am_{22} & \cdots & am_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ am_{n1} & am_{n2} & \cdots & am_{nk}\end{pmatrix}\end{array}$$

where, $am_{ij}\in[0,1]$ is the importance degree of ith criterion from the viewpoint of jth DM. Then, similar to afore part, we apply CI aggregator again to aggregate each row of AM matrix to a single value as the weight of respective criterion:

$$cd_i=(C)\int am_{i0}d\mu=\sum_{j=1}^{k}[am_{ij}^*-am_{ij-1}^*]\mu\{c_j^*,c_{j+1}^*,\cdots,c_k^*\},$$

where, $am_{i1}^*\le am_{i2}^*\le\cdots\le am_{ik}^*$, cd_i is the total weight of ith criterion and $CD=(cd_1,cd_2,\cdots,cd_n)$ is the weight vector of criteria that applied by DMs in ranking process of alternatives.

7 Solving an MAGDM problem using PA operator and Choquet integral

By combining PA operator and CI, we will propose a new way for solving an MAGDM problem in this section.
Let $A=\{A_1,A_2,\cdots,A_n\}$ be n alternatives to be ranked against all attributes/criteria set $C=\{c_1,c_2,\cdots,c_m\}$ with weight vector $WC=(wc_1,wc_2,\cdots,wc_n)$ which, is determined by the manager.

To do it, the manager invites a group of DMs as $D=\{D_1,D_2,\cdots,D_k\}$ which, according to the variety of their scientific expertise and individual differences, have different importance as weight vector $WD=(wd_1,wd_2,\cdots,wd_k)$. Based on the previous Section, based on the importance degree of criteria are defined by the manager i.e. WC, we proposed to determine the importance degree of each DMs (WD), firstly. Next, we apply WD to compute the weight vector CD. At this time, the alternatives are evaluated up to all criteria by each DMs, i.e., there exist k decision matrices $D^P=[\tilde{d}_{ij}^P]_{n\times m}$, $p=1,2,\cdots,k$, which $\tilde{d}_{ij}^P$ is appraisement of pth DM from ith alternative against to jth attribute.

In order to rank the alternatives, we proposed to aggregate the decision matrices to a single matrix using the proposed power average aggregator in this paper, firstly. Then, we apply Choquet integral on each row of the aggregated decision matrix to compute the score of

alternatives. Finally, the ranking of alternatives is in accordance with the ranking of their scores.

The proposed method

This method will be expressed in three steps:
Let the DM's assessments are assumed to be trapezoidal IT2FNs, i.e.

$$\tilde{d}_{ij}^{P} = (d_{ij}^{PU}, d_{ij}^{PL}) = \langle (d_{ij1}^{PU}, d_{ij2}^{PU}, d_{ij3}^{PU}, d_{ij4}^{PU}; 1), (d_{ij1}^{PL}, d_{ij2}^{PL}, d_{ij3}^{PL}, d_{ij4}^{PL}; h) \rangle,$$

where, $h \in [0,1]$.
Step one: Construct the aggregated decision matrix through the decision matrices $D^{P} = [\tilde{d}_{ij}^{P}]_{n\times m}$, $p = 1, 2, \cdots, k$, and weight vector $WD = (wd_1, wd_2, \cdots, wd_k)$, using the weighted PA operator i.e., $D = [\tilde{d}_{ij}]_{n\times m}$, where $\tilde{d}_{ij} = PAIT2FN_w(\tilde{d}_{ij}^{1}, \tilde{d}_{ij}^{2}, \cdots, \tilde{d}_{ij}^{k})$.
Step two: Get $CD = (cd_1, cd_2, \cdots, cd_n)$ as entrance vector and compute the joint measure of each nonempty subset of its element using λ-fuzzy concept.
Step three: aggregate all rows of aggregated decision matrix $D = [\tilde{d}_{ij}]_{n\times m}$ by CI i.e.:

$$\tilde{A}_i = (C)\int \tilde{d}_{i0} d\mu = \sum_{j=1}^{n} [\tilde{d}_{ij}^{*} - \tilde{d}_{ij-1}^{*}].\mu(\{c_i^{*}, c_{i+1}^{*}, \cdots, c_n^{*}\}),$$

where, $\tilde{d}_{i0}^{*} = \langle (0,0,0,0;1), (0,0,0,0;1) \rangle$, $\mu(\{c_i^{*}, c_{i+1}^{*}, \cdots, c_n^{*}\})$ is a joint measure of attribute $\{c_i^{*}, c_{i+1}^{*}, \cdots, c_n^{*}\}$ and $\{c_1^{*}, c_2^{*}, \cdots, c_n^{*}\}$ is permutation of $\{c_1, c_2, \cdots, c_n\}$ such that

$$\tilde{d}_{i1}^{*} \le \tilde{d}_{i2}^{*} \le \ldots \le \tilde{d}_{ij}^{*}.$$

Step four: Rank the computed Choquet integral values $\tilde{A}_i$ and apply it as a ranking order of alternatives.

8 Numerical example

We'll apply the proposed method to solve an MAGDM problem.
Example Let $\{A_1, A_2, A_3\}$ be the alternatives to be ranked based on the criteria set $\{c_1, c_2, c_3\}$. It will be done by a group of DMs $\{D_1, D_2, D_3\}$. Let $WC = (0.3, 0.6, 0.4)$ be the weight vector of the criteria set obtained by the direct question from the manager. The relation $\sum_{i=1}^{3} w_i \ne 1$ implied that the criteria are interactive and we have to use CI in the aggregation process of each row of weight matrix WM and then the DM's importance degrees, are denoted by weighting vector $WD = (wd_1, wd_2, wd_3)$.

Table 1 Measures of all subset of criteria set C

Fuzzy-measure	Quantity	Fuzzy-measure	Quantity
$\mu(\{c_1, c_2\})$	0.7912	$\mu(\{c_1, c_3\})$	0.8275
$\mu(\{c_2, c_3\})$	0.8550	$\mu(\{c_1, c_2, c_3\})$	1

Let the DMs are rated from the perspective of each criterion by manager and construct WM matrix as following:

$$WM = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{pmatrix} 0.5 & 0.7 & 0.2 \\ 0.3 & 0.2 & 0.8 \\ 0.9 & 0.4 & 0.5 \end{pmatrix} \end{array}$$

Now, we have to compute the weight of each non-empty subset of criteria set. It is done by λ-fuzzy measure with $\lambda = -0.6042$ and displayed in Table 1. It helps us to aggregate WM 's rows using CI as follows:

$$wd_1 = (0.2-0)\times 1 + (0.5-0.2)\times o.7912 + (0.7-0.5)\times 0.6 = 0.5574,$$
$$wd_2 = (0.2-0)\times 1 + (0.3-0.2)\times o.8257 + (0.8-0.3)\times 0.4 = 0.4826,$$
$$wd_3 = (0.4-0)\times 1 + (0.5-0.4)\times o.8275 + (0.9-0.5)\times 0.3 = 0.5726.$$

Table 2 Measure of all subset of DMs

Fuzzy-measure	Quantity	Fuzzy-measure	Quantity
$\mu(\{D_1, D_2\})$	0.8146	$\mu(\{D_1, D_3\})$	0.8631
$\mu(\{D_2, D_3\})$	0.8253	$\mu(\{D_1, D_2, D_3\})$	1

Then, we have $WD = (0.5574, 0.4826, 0.5726)$ as the weights of decision makers and are used to specify the weights of criteria, from the standpoint of DMs. Let assessment matrix is completed as follows:

$$AM = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{pmatrix} 0.6 & 0.7 & 0.4 \\ 0.9 & 0.3 & 0.8 \\ 0.5 & 0.4 & 0.8 \end{pmatrix} \end{array}$$

Then, due to weight vector $WD = (0.5574, 0.4826, 0.5726)$ each row of AM matrix will be aggregated to a single value, using CI with $\lambda = -0.8319$ and according to the given values in Table 2.

So, we have $CD = (0.6202, 0.7873, 0.6581)$ as the new weight vector of criteria that, is used in ranking order of alternatives. Due to the use of these values in integration process, it is necessary to calculate the weight of their combinations as in Table 3.

$$cd_1 = (0.4-0)\times 1 + (0.6-0.4)\times 0.8146 + (0.7-0.6)\times 0.5726 = 0.6202,$$
$$cd_2 = (0.3-0)\times 1 + (0.8-0.3)\times 0.8631 + (0.9-0.8)\times 0.5574 = 0.7873,$$
$$cd_3 = (0.4-0)\times 1 + (0.5-0.4)\times 0.8631 + (0.8-0.5)\times 0.5726 = 0.6581.$$

Table 3 Measure of each all of criteria set C

Fuzzy-measure	Quantity	Fuzzy-measure	Quantity
$\mu(\{cd_1, cd_2\})$	0.9365	$\mu(\{cd_1, cd_3\})$	0.8846
$\mu(\{cd_2, cd_3\})$	0.9456	$\mu(\{cd_1, cd_2, cd_3\})$	1

Each DM evaluates the alternatives against to all criteria and arranged his/her opinions in a matrix. Thus,

$$\tilde{D}^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} c_1 \\ \left(\begin{array}{c} \langle(4,5,6,7;1),(4.5,5,6,6.5;.8)\rangle \\ \langle(2,4,5,7;1),(3.0,4,5,6.0;.7)\rangle \\ \langle(6,7,8,9;1),(6.5,7,8,8.5;.8)\rangle \end{array} \right. \end{array}$$

$$\begin{array}{c} c_2 \\ \langle(1,5,6,9;1),(3.0,4,5.0,6;.6)\rangle \\ \langle(5,6,7,8;1),(5.5,6,6.5,7;.9)\rangle \\ \langle(2,4,5,7;1),(3.0,4,5.0,6;.7)\rangle \end{array}$$

$$\begin{array}{c} c_3 \\ \langle(6,7,8,9;1),(6.5,7,8,9;.8)\rangle \\ \langle(1,5,6,9;1),(3.0,4,5,6;.6)\rangle \\ \langle(1,5,6,9;1),(3.0,4,5,6;.6)\rangle \end{array} \Bigg)$$

$$\tilde{D}^2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} c_1 \\ \left(\begin{array}{c} \langle(1,5,6,9;1),(3.0,4,5,6.0;.6)\rangle \\ \langle(2,4,5,7;1),(3.0,4,5,6.0;.7)\rangle \\ \langle(6,7,8,9;1),(6.5,7,8,8.5;.8)\rangle \end{array} \right. \end{array}$$

$$\begin{array}{c} c_2 \\ \langle(1,5,6,9;1),(3.0,4,5.0,6;.6)\rangle \\ \langle(6,7,8,9;1),(6.5,7,8,8.5;.8)\rangle \\ \langle(3,5,7,9;1),(4,5.5,6.5,8;.9)\rangle \end{array}$$

$$\begin{array}{c} c_3 \\ \langle(5,6,7,8;1),(5.5,6,6.5,7;.9)\rangle \\ \langle(6,7,8,9;1),(6.5,7,8.0,9;.8)\rangle \\ \langle(5,6,7,8;1),(5.5,6,6.5,7;.9)\rangle \end{array} \Bigg)$$

$$\tilde{D}^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} c_1 \\ \left(\begin{array}{c} \langle(2,4,5,7;1),(3.0,4,5,6.0;.7)\rangle \\ \langle(1,5,6,9;1),(3.0,4,5,6.0;.6)\rangle \\ \langle(6,7,8,9;1),(6.5,7,8,8.5;.8)\rangle \end{array} \right. \end{array}$$

$$\begin{array}{c} c_2 \\ \langle(3,5,7,9;1),(4,5.5,6.5,8;.9)\rangle \\ \langle(5,6,7,8;1),(5.5,6,6.5,7;.9)\rangle \\ \langle(6,7,8,9;1),(6.5,7,8.0,9;.8)\rangle \end{array}$$

$$\begin{array}{c} c_3 \\ \langle(6,7,8,9;1),(6.5,7,8.0,9;.8)\rangle \\ \langle(1,5,6,9;1),(3.0,4,5.0,6;.6)\rangle \\ \langle(3,5,7,9;1),(4,5.5,6.5,8;.9)\rangle \end{array} \Bigg).$$

To solve this problem, $\tilde{D}^1, \tilde{D}^2$ and $\tilde{D}^3$ as decision matrices, using PA operator, will be aggregated to a single decision matrix $\tilde{D}$, firstly. Then we have:

$$l\tilde{D} = \left(\begin{array}{l} \langle(2.3,4.7,5.7,7.7;1),(3.5,4.3,5.4,6.2;0.6)\rangle \\ \langle(1.7,4.3,5.3,7.6;1),(3.0,4.0,5.0,6.0;0.7)\rangle \\ \langle(6.0,7.0,8.0,9.0;1),(6.5,7.0,8.0,8.5;0.8)\rangle \end{array} \right.$$

$$\begin{array}{l} \langle(1.6,5.0,6.3,9.0;1),(3.3,4.5,5.5,6.6;0.6)\rangle \\ \langle(4.0,4.8,5.6,6.3;1),(4.4,4.8,5.3,5.7;0.8)\rangle \\ \langle(2.2,3.6,4.7,6.1;1),(3.0,4.0,4.6,5.5;0.7)\rangle \end{array}$$

$$\left. \begin{array}{l} \langle(5.7,6.7,7.7,8.7;1),(6.2,6.7,7.5,8.4;0.8)\rangle\rangle \\ \langle(1.6,5.3,6.3,9.0;1),(3.4,4.4,5.4,6.4;0.6)\rangle \\ \langle(4.7,6.0,7.3,8.6;1),(5.3,6.1,7.0,8.0;0.8)\rangle \end{array} \right).$$

Now, we aggregate each row of decision matrix $\tilde{D}$ to a single value, using Choquet integral and values given in Table 3:

$$(C)\int D^1 d\mu = \langle(4.4,6.0,7.0,8.5;1),(5.3,5.9,6.8,7.7;0.6)\rangle,$$
$$(C)\int D^2 d\mu = \langle(3.2,4.9,5.8,7.1,1),(4.0,4.6,5.3,5.9;0.6)\rangle,$$
$$(C)\int D^3 d\mu = \langle(5.2,6.3,7.4,8.5;1),(5.8,6.4,7.3,8;0.7)\rangle.$$

Finally, these values which, interpreted as the scores of alternatives, are ranked to obtain the ranking order of alternatives:

$$A_2 < A_1 < A_3.$$

If we solved it using CI aggregator, we would obtain $A_1 < A_2 < A_3$. In [40] which, CI is combined with TOPSIS method a similar ranking order as $A_2 < A_1 < A_3$ is obtained. It is

shown that our proposed method extracts more information from uncertain situations and then we are closer to the optimal decision. It is necessary to note that this method doesn't have the complexity of the previous method ([40]).

9 Conclusion

Undoubtedly, dealing with uncertain information is increasingly growing. IT2FNs can help us to have a more logical use of such data. In this paper, we proposed a new ranking method for IT2FNs, and aggregate them using CI of IT2FNs, and PA of IT2FNs, separately. Also, we offered a new weighting method based on CI. The proposed methods are used to solve an FMAGDM problem.

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