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A new four-parameter distribution: properties and applications

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Abstract In this paper a new four-parameter lifetime distribution named "the Exponentiated gompertz-poisson (E-GP) distribution" has been suggested that it has a decreasing, increasing, bathtub-shaped and inverse bathtub-shape for modeling lifetime data. The Exponentiated gompertz-poisson has applications in survival, reliability modeling, lifetime and queuing problems. Firstly, the mathematical and statistical characteristics of the proposed distribution are presented, and then the applications of the new distribution are studied using the real data set. Its first moment about origin and moments about mean have been obtained and expressions for skewness, kurtosis have been given. Various mathematical and statistical properties of the proposed distribution have been discussed. Estimation of its parameter has been discussed using the method of maximum likelihood. In the end, two applications of the new distribution have been discussed with two real lifetime data sets. The results also confirmed the suitability of the presented models for real data collection.

Keyword: Gompertz-Poisson Distribution (GP), Exponentiated Gompertz-Poisson Distribution (EGP), Moments, Lifetime Data, Parameter Estimation and Goodness of Fit.

1 Introduction

Modeling lifetime data is an important subject in areas like engineering, medicine, and finance. Marshall and Olkin [11] introduced a new method by combining a continuous and a discrete distribution and used non-negative integer. Several articles have used this method and introduced new distributions. Marshall and Olkin generalized the Weibull and exponential distributions. Alice and Juss [3] introduced the half-parametric Marshall and Olkin distribution. Ghitany et al. [9] also Introduced Marshall-Olkin Weibull and examined properties by using censored data. Ghitany et al. [8] introduced the generalized Lomax distribution of Marshall olkin. Also, some authors introduced some new distributions by combining of Poisson and exponential distribution. Kus [10] introduced The Poisson exponential distribution that has decreasing hazard rate function and Cancho et al. [5] introduced the generalization of the Poisson exponential distribution and Al-Awadhi [2] introduced The Poisson lomax distributions.

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Gompertz's distribution plays an important role in modeling and analyzing mortality data (Wetterstrand [14], Gavlirov [7]). The hazard rate function of this distribution is increasing, so, it can't be used to describe the states that the data follows the distributions with one-mode hazard rate function. Later some researches introduced the Gompertz-Poisson distribution by combining the Gompertz and Poisson distributions cut at zero that its hazard rate function can be decreasing, increasing, bathtub and inverse bathtub-shaped depending on the values of the parameters.

The gompertz-poisson distribution is used in business, economics, actuarial and biological sciences.

The probability density function of the gompertz-poisson distribution was presented as follows:

$$f(x)=\frac{\lambda\alpha e^{\beta x} e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)} e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}}{1-e^{-\lambda}}, x>0, \lambda, \alpha>0, \beta \in \ (-\infty,+\infty) \qquad (1)$$

with shape parameter λ and scale parameter β and the survival function (sf) associated to (1) is:

$$S(x)=1-\frac{e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}-e^{-\lambda}}{1-e^{-\lambda}} \qquad (2)$$

The purpose of this paper is to provide a new four-parameter distribution that can be established by the relationship (1) and has increasing, decreasing, bathtub and invers bathtub hrfs and it is very flexible model in lifetime models. This distribution in the same conditions works better than the other distributions for modeling lifetime data. We call this new distribution “(E-GP) ”.

2 The new model and its properties

A random variable X has the gompertz-poisson distribution with three parameters if its cumulative distribution function (cdf) is given by:

$$F(x)=\frac{e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}-e^{-\lambda}}{1-e^{-\lambda}}, x \geq -\theta,\ \ \theta, \lambda>0, \quad \beta \in (-\infty,+\infty) \qquad (3)$$

That θ and β are the scale and λ is the shape parameters.

Now suppose that the random variable *X* has the cdf[†] (3), then its exponentiated distribution function, which is the cdf of E-GP[‡] distribution, is:

$$F(x)=\left[\frac{e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}-e^{-\lambda}}{1-e^{-\lambda}}\right]^{\gamma};\ \ x>-\theta, \lambda, \theta, \gamma>0, \beta \in (-\infty,+\infty) \qquad (4)$$

The pdf[§] of the E-GP distribution is obtained by differentiating (4) and therefore we have:

[†] Cumulative distribution function
[‡] Exponentiated Rayleigh Lomax

$$f(x)=\gamma\left(\frac{\lambda\alpha e^{\beta x}e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}}{1-e^{-\lambda}}\right)\left[\frac{e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}-e^{-\lambda}}{1-e^{-\lambda}}\right]^{\gamma-1};$$

$$x>-\theta,\lambda,\theta,\gamma>0,\beta\in(-\infty,+\infty) \tag{5}$$

The hrf of the new distribution is given by:

$$h(x)=\frac{\gamma\left(\frac{\lambda\alpha e^{\beta x}e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}}{1-e^{-\lambda}}\right)\left[\frac{e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}-e^{-\lambda}}{1-e^{-\lambda}}\right]^{\gamma-1}}{1-\left[\frac{e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}-e^{-\lambda}}{1-e^{-\lambda}}\right]^{\gamma}} \tag{6}$$

The pdfs of the new distribution are plotted in Figure 1 for some selected values of parameters. It can be seen that the pdf of this distribution is flexible. Figure 2 contains the plots of the hrfs of the E-GP distribution for different values of parameters. From Figure 2, we can observe that the hrf is increasing, decreasing, bathtub and invers bathtub depending on the parameter values.

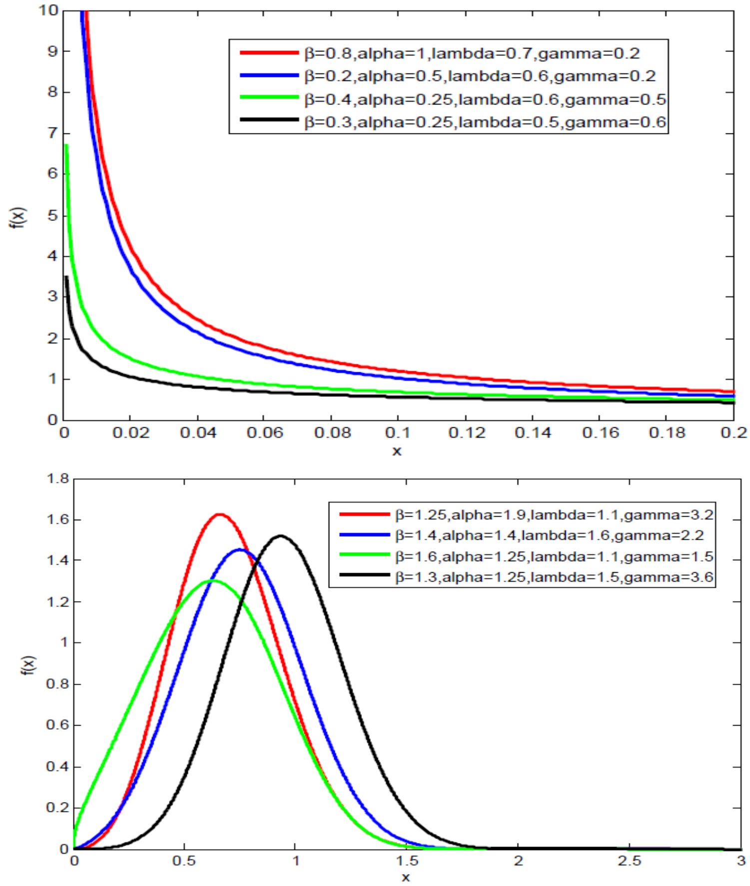
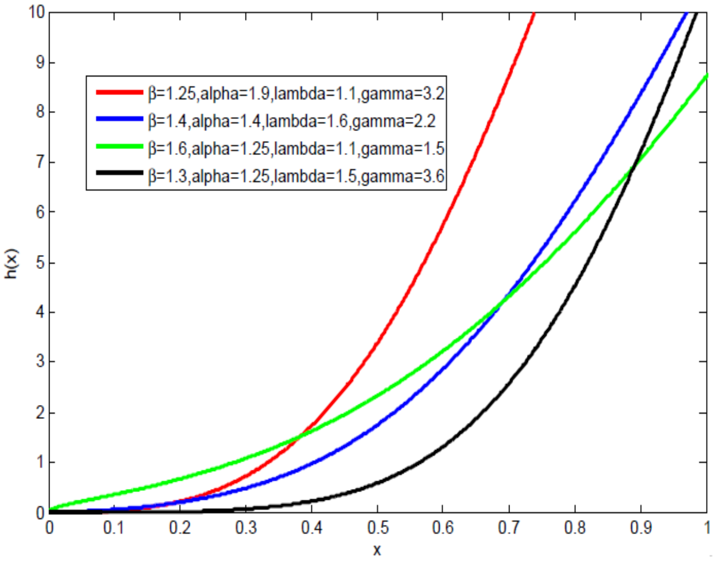


Fig. 1 Pdfs of the E-GP distribution for some selected values of α and β.

§ probability density function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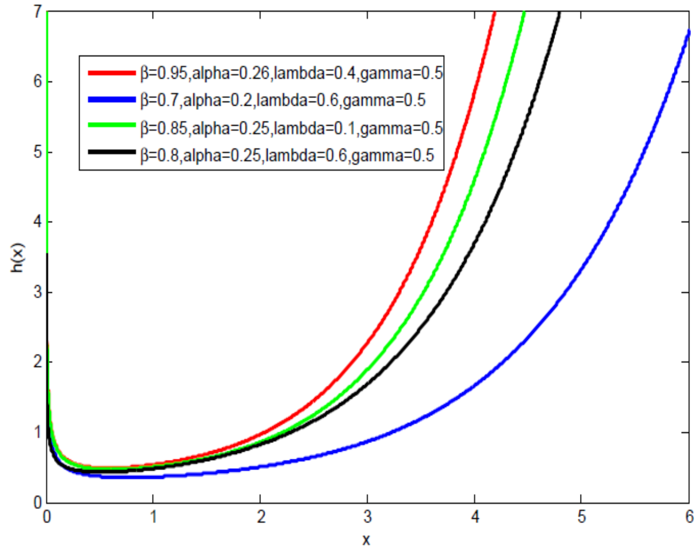
β=1.4,alpha=1.4,lambda=1.6,gamma=2.2
β=1.6,alpha=1.25,lambda=1.1,gamma=1.5
β=1.3,alpha=1.25,lambda=1.5,gamma=3.6
h(x)
x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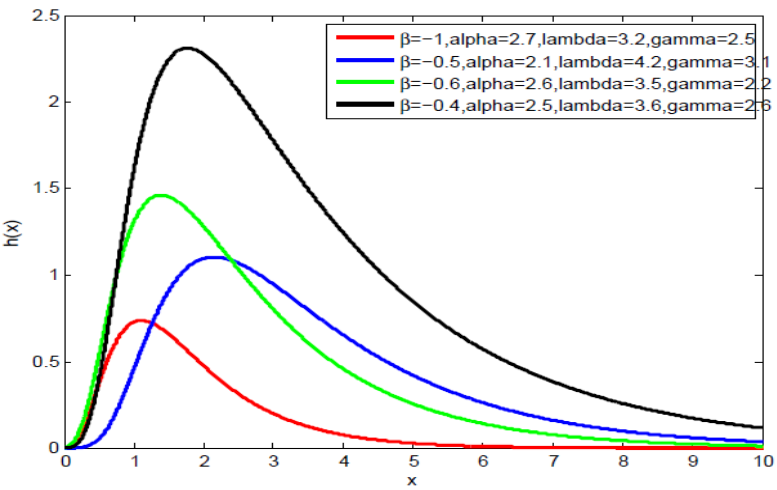
β=0.7,alpha=0.2,lambda=0.6,gamma=0.5
β=0.85,alpha=0.25,lambda=0.1,gamma=0.5
β=0.8,alpha=0.25,lambda=0.6,gamma=0.5
h(x)
x


β=−1,alpha=2.7,lambda=3.2,gamma=2.5
β=−0.5,alpha=2.1,lambda=4.2,gamma=3.1
β=−0.6,alpha=2.6,lambda=3.5,gamma=2.2
β=−0.4,alpha=2.5,lambda=3.6,gamma=2.6
h(x)
x

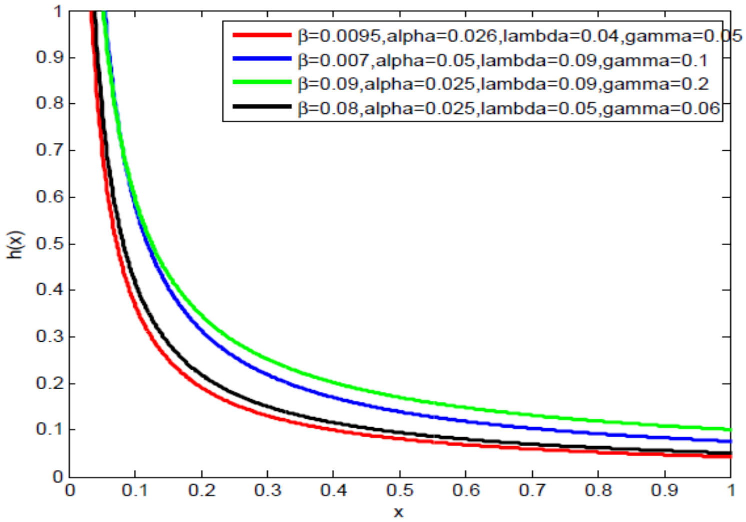


Fig. 2 Hrfs of the E-GP distribution for some selected values of α and β.

3 The moments and incomplete moments of the new distribution

Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution. In this section, we present complete and incomplete moments of the E-GP distribution. But first, we present an expansion for $f(x)$ in order to obtain expressions for the moments. Using the binomial expansion and the Expansion of $e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}$, we have:

$$f(x)=\gamma\left(\frac{\lambda\alpha e^{\beta x} e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)} e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}}{1-e^{-\lambda}}\right)\left[\frac{e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}-e^{-\lambda}}{1-e^{-\lambda}}\right]^{\gamma-1}$$

$$=\frac{\gamma\lambda\alpha e^{\beta x} e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1-e^{-\lambda})^{\gamma}}\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^n e^{\frac{-n\alpha}{\beta}\left(e^{\beta x}-1\right)}}{n!}\sum_{x=0}^{\gamma-1}\binom{\gamma-1}{x}\left(e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}\right)^{x}(-e^{-\lambda})^{\gamma-1-x}$$

$$=\frac{\gamma\lambda\alpha e^{\beta x} e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1-e^{-\lambda})^{\gamma}}\sum_{n=0}^{\infty}\sum_{x=0}^{\gamma-1}\frac{(-1)^n\lambda^n e^{\frac{-n\alpha}{\beta}\left(e^{\beta x}-1\right)}}{n!}\binom{\gamma-1}{x}\left(e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}\right)^{x}(-e^{-\lambda})^{\gamma-1-x} \tag{7}$$

So, the moment generating function of this distribution using (7) is obtained as follows:

$$M(x)=\int_0^{\infty} e^{tx} f(x)\,dx$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^{n}\lambda^{n}}{n!}\binom{n}{j}(-e^{-\lambda})^{n-j}\int_{0}^{\infty}e^{tx}\,e^{\frac{-n\alpha}{\beta}\left(e^{\beta x}-1\right)}\,\left(e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}\right)^{j}e^{\beta x}e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}\,dx \tag{8}$$

There are many softwares such as MATHEMATICA, MATLAB and MAPLE can be used to compute (8) numerically.
The *r*-th moment of the E-GP distribution is obtained by using the following formula:

$$E[X^{r}]=\int_{0}^{\infty}x^{r}f(X)dx$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^{n}\lambda^{n}}{n!}\binom{n}{j}(-e^{-\lambda})^{n-j}\int_{0}^{\infty}x^{r}\,e^{\frac{-n\alpha}{\beta}\left(e^{\beta x}-1\right)}\,\left(e^{-\lambda e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}}\right)^{j}e^{\beta x}e^{\frac{-\alpha}{\beta}\left(e^{\beta x}-1\right)}\,dx \tag{9}$$

it can be calculated by MAPLE.

Based on the first four moments, the skewness and kurtosis of the new distribution can be obtained from the following equations respectively:

$$S=E[(X-E(X))^{3}]/(E[(X-E(X))^{2}])^{3/2}=(\mu_{3}-3\mu_{1}\mu_{2}+2\mu_{1}^{3})/[\mu_{2}-\mu_{1}^{2}]^{3/2},$$

and

$$K=E[(X-E(X))^{4}]/(E[(X-E(X))^{2}])^{2}=(\mu_{4}-4\mu_{1}\mu_{3}+6\mu_{1}^{2}\mu_{2}-3\mu_{1}^{4})/[\mu_{2}-\mu_{1}^{2}]^{2}.$$

where μ_r is the *r*-th moment of the E-GP distribution given in (9).

4 Maximum Likelihood Estimation

Let $x_1,\dots,x_n$ be a random sample of size *n* from the E-GP distribution. Then, the likelihood function is given by:

$$\mathrm{L}(x|\alpha,\beta,\lambda,\gamma)=\left(\frac{\gamma\alpha\lambda}{(1-e^{-\lambda})^{\gamma}}\right)^{n}e^{-\frac{\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x}-1\right)}\,e^{-\lambda e^{\frac{-\alpha}{\beta}}\sum_{i=1}^{n}\left(e^{\beta x}-1\right)}\,e^{\beta\sum_{i=1}^{n}x_i}\left[e^{-\lambda e^{\frac{-\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x}-1\right)}}-e^{-\lambda}\right]^{\gamma-1}$$

Consequently, the log-likelihood function is:

$$l\,(x|\alpha,\beta,\lambda,\gamma)=nlog\gamma+nlog\alpha+nlog\lambda-n\gamma\log(1-e^{-\lambda})-\frac{\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x}-1\right)$$
$$-\lambda e^{\frac{-\alpha}{\beta}}\sum_{i=1}^{n}\left(e^{\beta x}-1\right)+\beta\sum_{i=1}^{n}x_i-(\gamma-1)\left(\lambda e^{\frac{-\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x}-1\right)}-e^{-\lambda}\right)$$

The maximum likelihood estimates of the parameters may be obtained by maximizing the log-likelihood function with respect to the parameters. To this end, we take the derivatives of the log-likelihood function with respect to the parameters and then equate the results with zero. Therefore, the maximum likelihood estimates of β, λ, θ *and* γ, denoted by $\hat{\beta}, \hat{\lambda}, \hat{\theta}$ and $\hat{\gamma}$, respectively, are obtained by solving the following nonlinear equations simultaneously:

$$\frac{\partial \text{Ln L}(\alpha,\beta,\lambda,\gamma)}{\partial \alpha} = \frac{n}{\alpha} - \frac{1}{\beta}\sum_{i=1}^{n}\left(e^{\beta x} - 1\right)$$

$$+\frac{\lambda}{\beta}e^{\frac{-\alpha}{\beta}}\sum_{i=1}^{n}\left(e^{\beta x} - 1\right) - \lambda(\gamma - 1)\left(-\frac{1}{\beta}\sum_{i=1}^{n}\left(e^{\beta x} - 1\right)\right)\left(\lambda e^{\frac{-\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x}-1\right)}\right) = 0\,,$$

$$\frac{\partial \text{Ln L}(\alpha,\beta,\lambda,\gamma)}{\partial \beta}$$

$$= \frac{\alpha}{\beta^2}\sum_{i=1}^{n}\left(e^{\beta x} - 1\right) - \frac{\alpha}{\beta}\sum_{i=1}^{n}\left(\beta e^{\beta x}\right) - \lambda\left(\left(\frac{\alpha}{\beta^2}e^{\frac{-\alpha}{\beta}}\sum_{i=1}^{n}\left(e^{\beta x} - 1\right)\right) + e^{\frac{-\alpha}{\beta}}\sum_{i=1}^{n}\left(\beta e^{\beta x}\right)\right) + \sum_{i=1}^{n} x_i$$

$$-(\gamma - 1) \times \left(\lambda\left(\left(\frac{\alpha}{\beta^2}\right)\left(\sum_{i=1}^{n}\left(e^{\beta x} - 1\right)\right) + \sum_{i=1}^{n}\left(x e^{\beta x}\right)\left(\frac{-\alpha}{\beta}\right)\left(e^{\frac{-\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x}-1\right)}\right)\right)\right) = 0$$

$$\frac{\partial \text{Ln L}(\alpha,\beta,\lambda,\gamma)}{\partial \lambda} = \frac{n}{\lambda} - n\gamma\frac{e^{-\lambda}}{\left(1 - e^{-\lambda}\right)} - e^{\frac{-\alpha}{\beta}}\sum_{i=1}^{n}\left(e^{\beta x} - 1\right)$$

$$-(\gamma - 1)\left(e^{\frac{-\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x}-1\right)} + e^{-\lambda}\right) = 0$$

$$\frac{\partial \text{Ln L}(\alpha,\beta,\lambda,\gamma)}{\partial \gamma} = \frac{n}{\gamma} + n\log\left(1 - e^{-\lambda}\right) - \left(\lambda e^{\frac{-\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x}-1\right)} - e^{-\lambda}\right) = 0$$

The above equations do not seem to have explicit solutions, thus numerical methods may be applied to finding the roots.

In order to construct approximate confidence intervals and test hypotheses on the parameters, we obtain the observed Fisher information matrix, which is defined as:

$$I_n(\Theta) = -\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\lambda} & I_{\alpha\gamma} \\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\lambda} & I_{\beta\gamma} \\ I_{\lambda\alpha} & I_{\lambda\beta} & I_{\lambda\lambda} & I_{\lambda\gamma} \\ I_{\gamma\alpha} & I_{\gamma\beta} & I_{\gamma\lambda} & I_{\gamma\gamma} \end{bmatrix}$$

where $\Theta = (\alpha, \beta, \lambda, \gamma)^T$ is the vector of parameters and there are clear phrases for the observed information matrix elements.

$$I_{\alpha\alpha}=\frac{\partial^2 l}{\partial\beta^2} \quad ,I_{\alpha\beta}=\frac{\partial^2 l}{\partial\alpha\partial\beta} \quad ,I_{\alpha\lambda}=\frac{\partial^2 l}{\partial\alpha\partial\lambda} \quad ,I_{\alpha\gamma}=\frac{\partial^2 l}{\partial\alpha\partial\gamma}$$
$$I_{\beta\alpha}=\frac{\partial^2 l}{\partial\beta\partial\alpha} \quad ,I_{\beta\beta}=\frac{\partial^2 l}{\partial\beta^2} \quad ,I_{\beta\lambda}=\frac{\partial^2 l}{\partial\beta\partial\lambda} \quad ,I_{\beta\gamma}=\frac{\partial^2 l}{\partial\beta\partial\gamma}$$
$$I_{\lambda\alpha}=\frac{\partial^2 l}{\partial\lambda\partial\alpha} \quad ,I_{\lambda\beta}=\frac{\partial^2 l}{\partial\lambda\partial\beta} \quad ,I_{\lambda\lambda}=\frac{\partial^2 l}{\partial\lambda^2} \quad ,I_{\lambda\gamma}=\frac{\partial^2 l}{\partial\lambda\partial\gamma}$$
$$I_{\gamma\alpha}=\frac{\partial^2 l}{\partial\gamma\partial\alpha} \quad ,I_{\gamma\beta}=\frac{\partial^2 l}{\partial\gamma\partial\beta} \quad ,I_{\gamma\lambda}=\frac{\partial^2 l}{\partial\gamma\partial\lambda} \quad ,I_{\gamma\gamma}=\frac{\partial^2 l}{\partial\gamma^2}$$

Under the regularity conditions (see for example [6], Lehmann and Casella, 1998, pp. 461-463), the asymptotic inference for the vector of parameters, i.e $\Theta=(\alpha,\beta,\lambda,\gamma)^T$, based on normal approximation can be used. When the sample size n is large enough, then $\sqrt{n}(\hat{\theta}-\theta)$ is asymptotically a four-variate normal random vector with mean $(0,0)^T$and the variance-covariance matrix that equates to the inverse of the expected Fisher information matrix i.e. $\mathrm{J}(\theta)^{-1}$. This asymmetric behavior holds if we replace $\mathrm{J}(\theta)^{-1}$with $\left[\frac{1}{n}I_n(\theta)\right]^{-1}$. If unknown parameters appear in the variance-covariance matrix, then they can be replaced by their respective maximum likelihood estimates. Using the normal approximation, we can obtain approximate (asymptotic) confidence intervals for the parameters.

5 Applications

In this section, we present applications of E-GP distribution using real data. These applications demonstrate the flexibility of this distribution compared to the other models for the real data set. We compare the fit of the E-GP distribution with those of some other lifetime distributions which are gompertz-poisson, Exponentiated Weibull–Poisson and beta Genealized exponential distribution.

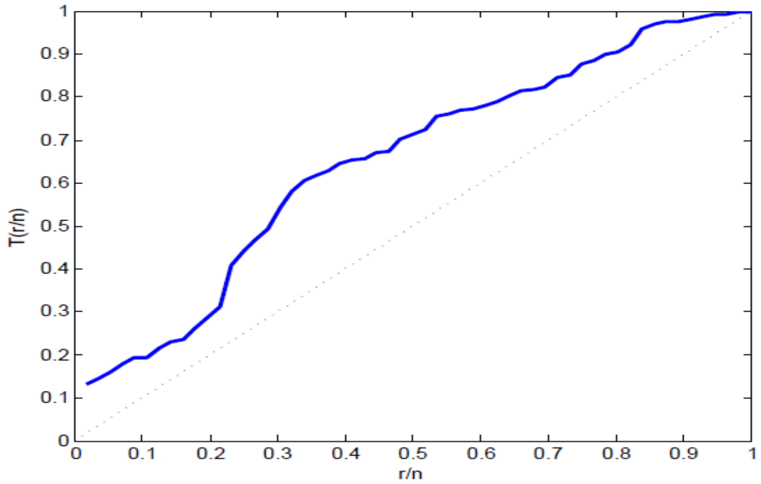
All the computations presented in this section were done using the MATLAB and R software. The first data set (Set I) are the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England [4]. Unfortunately, the units of measurement are not given in the paper. The data set is:

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64,1.68, 1.73, 1.81, 2 ,0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01,0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48,1.5, 1.55,1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67,1.7, 1.78, 1.89.

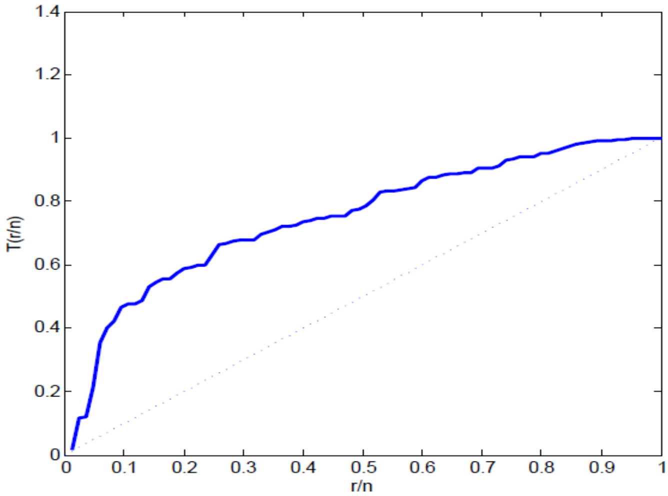
The second data set (Set II) is the original test records of the time-to-failure data for 40 suits of turbochargers that are taken from [15].

1.6, 2, 2.6, 3, 3.5, 3.9, 4.5, 4.6, 4.8, 5, 5.1, 5.3, 5.4, 5.6, 5.8, 6, 6, 6.1, 6.3, 6.5, 6.5, 6.7, 7, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9

In many applications there is qualitative information about the failure rate shape, which can help with selecting a particular model. It called the total time on test (TTT) plot, see Aarset [1]. The TTT plot is obtained by plotting $T({}^{r}/_{n}) = [(\sum_{i=}^{n} y_{i:n}) + (n-r)y_{r:n}]/\sum_{i=}^{n} y_{i:n}$ against ${}^{r}/_{n}$. It is a straight diagonal for constant failure rates. It is convex for decreasing failure rates and concave for increasing failure rates. It is first convex and then concave if the failure rate is bathtub-shaped. It is first concave and then convex if the failure rate is upside-down bathtub. The TTT plots for the two data sets are presented in Fig. 3(a) and (b), respectively. The TTT plot for the Set I and II indicates an increasing HR.



(a)



(b)

Fig. 3 Histogram, TTT plot for the Set I and Set II.

We use the Kolmogorov-Smirnov (K-S) test statistics, the Akaike information criterion (AIC) and the Bayesian criterion (BIC) in order to compare the fits. The computed MLEs, K-S test statistics and the values of AIC and BIC for both data sets are given in Table 1. These criteria are widely utilized to check how closely a specified cdf fits the empirical distribution of a given data set. It is well-known that the smaller values of AIC, BIC and K-S test statistic mean a better fit to the data. Here, it is observed from Table 1 that the E-GP model outperforms all the other considered models in the sense of the considered criteria.

Table 1 The maximum likelihood estimates of the parameters, K-S test and the values of AIC and BIC for real data.

DIST.	MLEs	K-S	AIC	BIC
E-GP	$\hat{\alpha} = 0.2454, \hat{\beta} = 1.5886, \hat{\lambda} = 3.3231, \hat{\gamma} = 1.5626,$	0.0921	30.56	47.89
GP	$\hat{\alpha} = 0.1021, \hat{\beta} = 2.3730, \hat{\lambda} = 3.6678$	0.1032	32.71	51.57
BGE	$\hat{\alpha} = 24.2342, \hat{\lambda} = 0.9499, \hat{a} = 0.3786, \hat{b} = 91.54,$	0.1725	39.25	64.39
EWP	$\hat{\alpha} = 0.5790, \hat{\beta} = 0.6466, \hat{\lambda} = 5.499, \hat{\theta} = 2.465,$	0.1166	34.52	59.27
E-GP	$\hat{\alpha} = 2.2526, \hat{\beta} = 1.2192, \hat{\lambda} = 3.1006, \hat{\gamma} = 1.7390,$	0.0643	163.98	178.65
GP	$\hat{\alpha} = 0.0085, \hat{\beta} = 0.6126, \hat{\lambda} = 0.3133$	0.0871	165.86	181.99
BGE	$\hat{\alpha} = 36.17, \hat{\lambda} = 0.1891, \hat{a} = 0.1280, \hat{b} = 1415.3,$	0.1039	167.62	189.13
EWP	$\hat{\alpha} = 0.0871, \hat{\beta} = 4.3125, \hat{\lambda} = 1.1674, \hat{\theta} = 0.2641,$	0.0964	166.34	184.65

It can be seen from the Table 1, the new distribution has the smallest values of the K-S test statistics, the smallest AIC and BIC values. Therefore, it can be concluded that the best fits belong to the new introduced model, i.e. the E-GP distribution among the other considered distribution in this section.

For the sake of visual comparison, the estimated pdfs of the considered distributions as well as the empirical histograms of the data sets are given in Figure 4. Also, the normal Q-Q plot for the data set I and II are plotted in Figure 5. It is obvious from the figures, that the proposed new distribution provides the best fit for real data set in comparison with the other considered distributions.

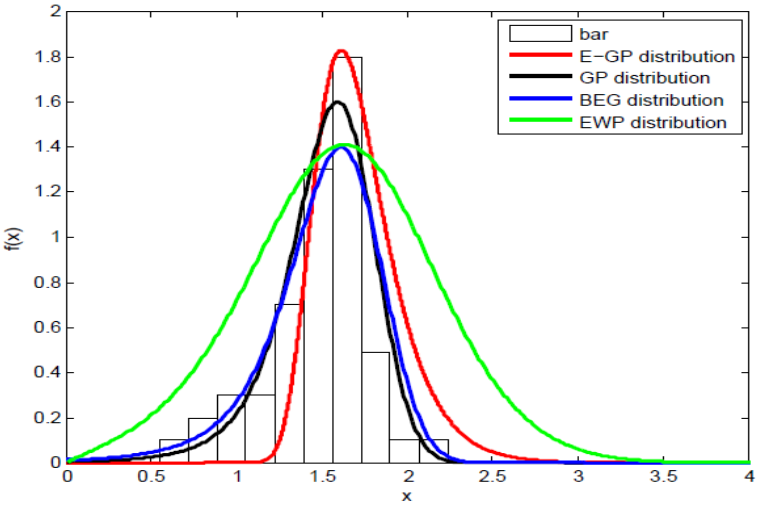
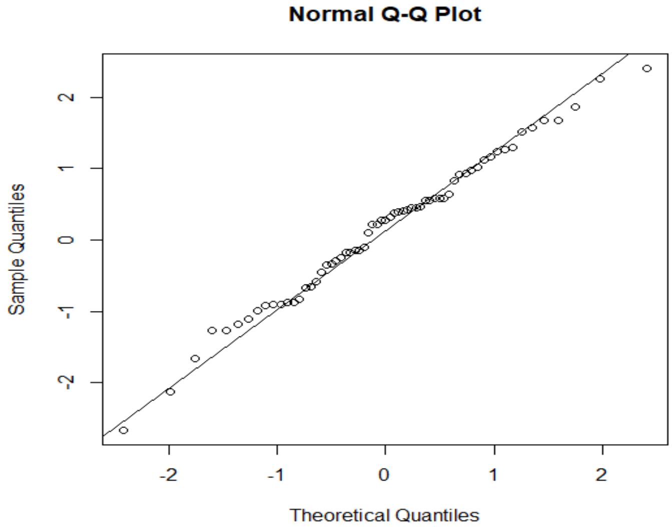


Fig. 4 the plot of the fitted probability density functions of considered distributions as well as the histogram for the data set I and the data set II.



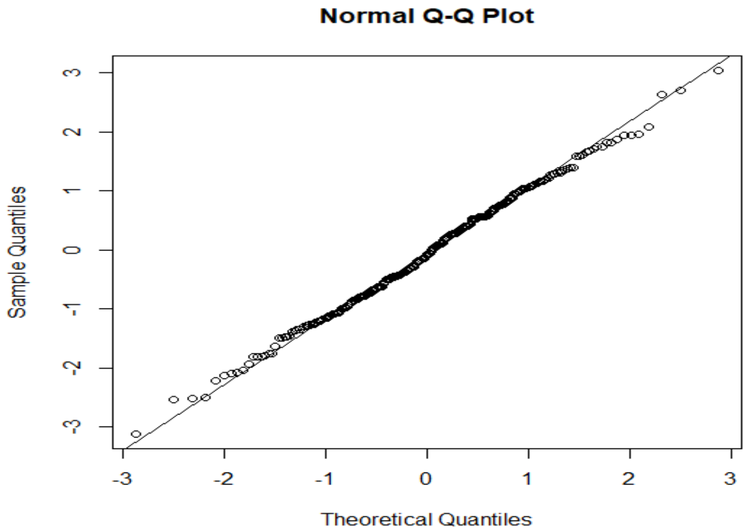


Fig. 5 the normal Q-Q plot for the data set I and II**.**

6 Discussion and conclusion

In this paper we provide a new four-parameter distribution that can be established by the relationship (1) and has increasing, decreasing, bathtub and invers bathtub hrfs and it is very flexible model in lifetime models. This distribution in the same conditions works better than the other distributions for modeling lifetime data.

7 Suggestions for future researchers

In this paper, we introduced a new distribution and worked on some of its mathematical properties. The new distribution has four parameters and its cdf and hrf have simple forms. The gompertz-poisson has decreasing, increasing, increasing-decreasing and unimodal hrf. The hrf of the new model can be decreasing, increasing, and inverse bathtub-shape depending on the values of the parameters so the new model is better than the other distributions for modeling lifetime data .To sum up, we can claim that the proposed distribution provides a quite flexible model for fitting many positive data sets that may appear in the areas like in survival, reliability modeling, lifetime and queuing problems and so on, in comparison with gompertz-poisson distribution. We note that the figure plotting and computations of this paper have been performed using R (R Core Team, 2017) [12]. Moreover, the package survival (see Therneau, 2015)[13] was used. The codes are available upon request.

There exist many other properties of the new distribution, like the reliability parameter, Kullback-Leibler divergence and so on, that have not been discussed in this paper. Moreover, some interesting inferential topics related to the new distribution are the Bayesian estimation of the parameters, prediction of future observations and estimation based on censored samples. We will probably report our findings regarding the mentioned subjects in the future.

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