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Grey prediction in linear programming problems

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Abstract The purpose of this paper is describes the use of grey pridiction in linear programming problems. Some definitions and concepts of grey system theory are introduced and then, we introduced GM(1,1) and fractional order accumulation into grey model. Due to the fluctuation of prices and the lack of certainty data in the market, optimal production was calculated to optimize the profit from sales using the grey prediction. The use of grey prediction models will be very useful in optimization problems and can provide decision-makers with an acceptable level. Prediction of the profit or cost of goods and services given the fluctuation of prices and, in some cases, the deficiency of data needed to make a decision can accurately reflect the power of predictive grey that uses very small data.

Keyword: Grey System Theory, Grey Linear Programming, GM(1,1), Grey Prediction.

1 Introduction

Forecasting the further system development directly from previous observation (data) seems to be a practicable alternative. In recent years, grey systems theory has been successfully employed in agriculture, industry, ecology, meteorology, earthquake, science and technology, medical care and other fields and grey theory of prediction is an important embracement of grey systems theory. GM(1,1) model is the main model in grey theory of prediction. It is one of the most widely used techniques in the grey systems. In recent years, it has also been successfully utilized in many fields and has demonstrated satisfactory results [3]. At present, grey forecast has been more widely used because it has advantages of a small sample data, computing convenience and short time forecast of high accuracy [13]. The forecasts will produce new information that will be happen in the future; this information is certainly beneficial for the business or policy makers and decision makers. But in terms of data collection is not any data is easily obtained so, the data are limited. The forecasts using grey forecasting model as one of the approaches that can be used to build a model with limited data sample, with forecasts of short-term problems, to generate forecasting models are valid and does not require consideration of the statistical distribution [12]. In these years, more scholars use the thought of fractional order accumulation into grey model. Wu Lifeng et al. [15] investigated a grey system model with the fractional order accumulation. Mao et al. [9]

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proposed Anovel fractional grey system model and its application. Some of the results were reported in various articles [16,18].

By combining the grey theory with the principle and method of linear programming problem, the linear programming model is established based on the grey theory. So far, many researchers have come up with different ways to solve grey linear programming. Li et al.[5] presented a Covered solution for a grey linear program based on a general formula for the inverse of a grey matrix. Chen et al. [1] and Liu et al. [6] developed some procedures to solve grey linear programming problems. Nasseri et al. [10] also provided a primal simplex algorithm for solving linear programming methods with grey objective function. The authers in [17] presented a grey linear programming model based on grey forcasting.

In this study, we used the GM(1,1) and fractional order accumulation into grey model. By using the grey prediction, gray linear programming with uncertain valu of price product solve and optimal production was calculated.

The rest of the paper is organized as follows: In Section 2, A brief overview of grey theory is given. In Section 3. The fundamentals of GM(1,1) model and model are introduced. In Section 4, grey linear programming problem was presented. An example is provided to clarify the approach in Section 5. Finally, conclusions are given in Section 6 briefly.

2 Grey system theory

Grey mathematics is the mathematical foundation of grey systems theory and its unit is a grey number [4]. In this section, some definitions and concepts which are useful in our further consideration about grey prediction and forecasting are introduced. Grey systems theory is one of the approaches used to study uncertainty, being superior in the mathematical analysis of systems with uncertain information [3]. Grey system theory, by which an information system can be classified into three categories, had been introduced for the first time by prof. Deng [11]. This theory includes white system, grey system and black system. If the system is completely unknown, it is called black while a system that is fully known is called white and a grey system is the system between black and white. Grey refers to incomplete information; in other words, information that is partially clear and partially unclear [14]. In summary, the main aim of grey system theory is to focus on the relation between analysis model structure and conditions such as uncertainty, multi data input, discrete data and lack of data for forecasting and decision making. Grey prediction uses GM(1,1) as a foundation for predicting existing data. In reality, this model seeks the future dynamic conditions of elements within a series.

Definition 1. A grey number is a number with clear upper and lower boundaries, but which has an unknown position within the boundaries. A grey number in the system is expressed mathematically as:

$$\otimes x \in [\underline{x}, \overline{x}] = \{t \in x \mid \underline{x} \leq t \leq \overline{x}\}$$

where $\otimes x$ is a grey number, t is information, $\underline{x}$ and $\overline{x}$ are the lower and upper limits of the information [2].

The transformation of an grey number to the appropriate crisp value can be made by using the whitening function, which can be shown as follows:

$\tilde{\otimes} \mathrm{a} = \alpha \mathrm{a} + (1-\alpha)\mathrm{b},\ \ \alpha \in [0,1]$ with α as whitening coefficient.

Definition 2. Suppose $\otimes x$ and $\otimes y$ are two grey numbers and $\otimes_1 \hat{x}$, $\otimes_2 \hat{x}$ are the kernel of $\otimes x$ and $\otimes y$, respectively, $g^{\circ}(\otimes_1 x)$ and $g^{\circ}(\otimes_2 x)$ are the degree of greyness of $\otimes x$ and $\otimes y$, respectively. So

if $\otimes \hat{x} < \otimes \hat{y}$, thus $\otimes x <_G \otimes y$;

if $\otimes \hat{x} = \otimes \hat{y}$, thus

(i) if $g^{\circ}(\otimes x) = g^{\circ}(\otimes y)$, thus $\otimes x =_G \otimes y$;

(ii) if $g^{\circ}(\otimes x) < g^{\circ}(\otimes y)$, thus $\otimes x >_G \otimes y$;

(iii) if $g^{\circ}(\otimes x) > g^{\circ}(\otimes y)$, thus $\otimes x <_G \otimes y$.

3 Grey Prediction Model

Grey prediction model investigates the preliminary data assisted by grey differential equation to extract the rules governing the system. The model creates a dynamic and continuous differential equation from the series of discrete data to actualize time-series prediction. Each grey model is expressed in the form of GM(*n*,*m*) wherein *n* is the order of the differential equation and *m* determines the number of variables.

3.1 GM(1,1) model

The Grey forecasting model GM(1,1) is a time series prediction model encompassing a group of differential equations adapted for parameter variance as well as a first-order differential equation.

In this section we focus on the grey prediction model, GM(1,1), which has been applied in many aspects of social and natural science, including decision-making, finance, economics, engineering and meteorology. GM(1,1) is the most applied models of time-series prediction model and it is basically an exponential model [7]. Liu and Deng studied the range suitable for GM(1,1) based on a simulated test. The area of validity, the area to be used carefully, the area not suitable for use and the prohibited area of GM(1,1) have been divided clearly according to the threshold of the developing coefficients [8]. In order to smooth the randomness, the primitive data obtained from the system to form the GM(1,1) is subjected to an operator, named Accumulating Generation Operator (AGO) [4]. The operator reveals the internal order pattern of the data or the trends of the data series. Then, the differential equation operationalizes system prediction in n stages. Finally, the prediction values and Inverse Accumulated Generating Operator (IAGO) are applied to figure out the main data estimates.

The procedure of GM (1, 1) grey prediction model can be summarized as follows.

Step 1. Let $x^{(0)}=(x^{(0)}(1), x^{(0)}(2),\ldots,x^{(0)}(n))$ denote a non-negative sequence of original data, where *n* is the length of the raw data sequence and $n \geq 4$.

Step 2. The new cumulative data sequence $x(1) = (x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n))$, which is the Accumulated Generating Operator (AGO) of $x^{(0)}$, is obtained as $x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i)$, k =1, 2, 3,…, *n*. First-order invers accumulated generating operator of $x^{(0)}$, is

$$\alpha^{(1)} X^{(0)}(k) = \left\{ \alpha^{(1)} x^{(0)}(2), \alpha^{(1)} x^{(0)}(3), \ldots, \alpha^{(1)} x^{(0)}(n) \right\} \tag{1}$$

Where

$$\alpha^{(1)} X^{(0)}(k) = x^{(0)}(k) - x^{(0)}(k-1); k = 1, 2, \ldots, n. \tag{2}$$

Step 3. The generated mean sequence $Z^{(1)}(k)$ of $x^{(1)}$ is defined as:

$$Z^{(1)} = \left\{ z^{(1)}(2), z^{(1)}(3), \ldots, z^{(1)}(n) \right\} \tag{3}$$

where $Z^{(1)}(k)$ is the mean value of adjacent data, i.e.

$$Z^{(1)}(k) = 0.5 x^{(1)}(k) + 0.5 x^{(1)}(k-1), \; k = 2, 3, \ldots, n. \tag{4}$$

The least square estimate sequence of the grey difference equation of GM(1,1) is defined as follows:

$$Z^{(1)}(k) = \alpha x^{(1)}(k) + (1-\alpha) x^{(1)}(k-1), \; k = 2, 3, \ldots, n \tag{5}$$

$$x^{(0)}(k) + a\, z^{(1)}(k) = b. \text{ (}\alpha \text{ is usually set at 0.5).} \tag{6}$$

Where a is the development coefficient and b is the input grey coefficient or grey parameter.

Step 4. Define the first-order differential equation of sequence $x^{(1)}$ as:

$$\frac{dx^{(1)}(t)}{dt} + a\, x^{(1)}(t) = b \tag{7}$$

In above mentioned equitation, t denotes the independent variables, a represents the grey developed coefficient of GM(1,1) model, and b is the grey controlled variable of the GM(1,1) model.

Step 5. Utilize the least squares estimation, we can derive the estimated first-order AGO sequence $x_p^{(1)}(k+1)$ and the estimated inversed AGO sequence $x_p^{(0)}(k+1)$ as follows,

$$x_p^{(1)}(k+1) = \left[x^{(0)}(1) - \frac{b}{a} \right] e^{-ak} + \frac{b}{a}. \tag{8}$$

$$x_p^{(0)}(k+1) = x_p^{(1)}(k+1) - x_p^{(1)}(k) \tag{9}$$

where k = 1, 2, 3, …, n.

parameters a and b can be conducted by the least square estimation methods as following equations:

$$[a, b]^T = \left[B^T B \right]^{-1} B^T y, \tag{10}$$

where $y = \left[x^{(0)}(2), x^{(0)}(3), \ldots, x^{(0)}(n) \right]^T$

$$B = \begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{bmatrix}.$$

To obtain the predicted value of the primitive data at time (k+1), IAGO is used to establish the following grey model:

$$x_p^{(0)}(k+1) = x_p^{(1)}(k+1) - x_p^{(1)}(k) = (x^{(0)}(1) - \frac{b}{a}) \left(1 - e^{a} \right) e^{-ak}, k = 1, 2, \ldots, n. \tag{11}$$

The predicted value of the primitive data at time (k + h) can be obtained as follows:

$$x_p^{(0)}(k+h) = (x^{(0)}(1) - \frac{b}{a}) \left(1 - e^{a} \right) e^{-a(k+h-1)}, k = 1, 2, \ldots, n. \tag{12}$$

In large data areas, the grey system prediction method based on small data mining as a new force suddenly rises, which becomes an effective tool for valuable information extraction from a mass of data. It is a very meaningful job to build more normal model testing standards based on the grey system prediction model testing method and statistical testing theory.

3.2 Fractional grey model

Grey model with the fractional order accumulation ($GM^{(\frac{p}{q})}(1,1)$) is new model. The procedure of $GM^{(\frac{p}{q})}(1,1)$ model can be summarized as follows [15,18]:

Step 1. For an original data sequence $x^{(0)}=(x^{(0)}(1), x^{(0)}(2),\ldots,x^{(0)}(n))$, $x^{(r)}$ is r-th order accumulated generating operation (r-AGO) of $x^{(0)}$, $x^{(r)}=(x^{(r)}(1), x^{(r)}(2),\ldots,x^{(r)}(n))$,

where $x^{(r)}(k)=\sum_{i=1}^{k} x^{(r-1)}(i)=\sum_{i=1}^{k} C_{k-i+r-1}^{k-i} x^{(0)}(i)$, k =1, 2, 3,…, *n*.

r-order invers accumulated generating operator of $x^{(0)}$ (r-IAGO) , is

$$\alpha^{(r)}X^{(0)}(k)=\left\{\alpha^{(r)}x^{(0)}(2),\alpha^{(r)}x^{(0)}(3),\ldots,\alpha^{(r)}x^{(0)}(n)\right\} \tag{13}$$

Where

$$\alpha^{(r)}X^{(0)}(k)=\alpha^{(r-1)}x^{(0)}(k)-\alpha^{(r-1)}x^{(0)}(k-1);\, k=1,2,\ldots,n. \tag{14}$$

If $r=\frac{p}{q}\,(0\le\frac{p}{q}\le 1)$, then $\frac{p}{q}$ -IAGO of $x^{(0)}$ is

$$\alpha^{(\frac{p}{q})}X^{(0)}(k)=\alpha^{(1)}X^{(1-\frac{p}{q})}(k)=\left\{\alpha^{(1)}X^{(1-\frac{p}{q})}(1),\alpha^{(1)}X^{(1-\frac{p}{q})}(2),\ldots\alpha^{(1)}X^{(1-\frac{p}{q})}(n)\right\} \tag{15}$$

Step 2. $Z^{(\frac{p}{q})}=\left\{z^{(\frac{p}{q})}(2),z^{(\frac{p}{q})}(3),\ldots,z^{(\frac{p}{q})}(n)\right\}$ is the mean generating sequence of the $\frac{p}{q}$ -AGO Sequence and, $Z^{(\frac{p}{q})}(k)=0.5x^{(\frac{p}{q})}(k)+0.5x^{(\frac{p}{q})}(k-1)$.

Step 3. Equation $X^{(\frac{p}{q})}(K)-X^{(\frac{p}{q})}(K-1)+aZ^{(\frac{p}{q})}(K)=b$ is $\frac{p}{q}$ -order GM(1,1) model.

Step 4. The least squares estimation for (a,b) of the $\frac{p}{q}$ -order GM(1,1) model satisfies

$$[a,b]^T=\left[B^TB\right]^{-1}B^Ty, \tag{16}$$

where

$$Y=\begin{bmatrix} x^{(\frac{p}{q})}(2)-x^{(\frac{p}{q})}(1)\\ x^{(\frac{p}{q})}(3)-x^{(\frac{p}{q})}(2)\\ \vdots\\ x^{(\frac{p}{q})}(n)-x^{(\frac{p}{q})}(n-1)\end{bmatrix}, B=\begin{bmatrix} -z^{(\frac{p}{q})}(2) & 1\\ -z^{(\frac{p}{q})}(3) & 1\\ \vdots & \vdots\\ -z^{(\frac{p}{q})}(n) & 1\end{bmatrix}. \tag{17}$$

Step 5. The whitenization equation $\frac{dx^{(\frac{p}{q})}(t)}{dt}+a\,x^{(\frac{p}{q})}(t)=b$ of $GM^{(\frac{p}{q})}(1,1)$ model is solved to obtain

$$x_p^{(1)}(t)=\left[x^{(0)}(1)-\frac{b}{a}\right]e^{-ak}+\frac{b}{a}. \tag{18}$$

4 Grey linear programming

We now define grey linear programming problems. Grey linear programming is a model of grey systems analysis for decision making under uncertainty. The field of grey linear programming has recently attracted significant interest. Linear programming problem with grey parameters is one of the convenient models of the well-known real problems as well as water resource planning, economics, geometry and etc. In this section, we define linear programming problems involving grey numbers as follows:

$$\max \otimes z =_G \otimes cx$$
$$s.t. \quad \otimes Ax \leq_G \otimes b,$$
$$x \geq 0.$$

where $x_j \in \mathbb{R}$ and $\otimes c_j, \otimes b_i, \otimes a_{ij} \in R(\otimes), i = 1,2,3,\ldots,m, \quad j = 1,2,3,\ldots,n.$

We call the above problem as a grey linear programming problem and it can be rewritten a $\max \otimes z =_G \otimes cx$ subject to $\otimes Ax \leq_G \otimes b,$ where $\otimes c$ is a $(1 \times n)$ grey vector, $\otimes b$ is an $(m \times 1)$ grey vector, and $\otimes A$ is an $(m \times n)$ grey matrix, and obviously all of them are consisting of grey numbers, and x is an unknown $(n \times 1)$ real vector.

In this section, we focus on linear programming problems such that they can be potentially used to make predictions.

5 Numerical example

In this section, for an illustration of the above approach, a numerical example of the linear programming problem which whitened model of grey linear programming will be solved based on the grey prediction method.

5.1 Example. A manufacturing company produces two products: Product A and Product B. Each piece of Product A requires 3 work days, 4 kilowatt-hours of electricity, and 9 tons of coals. And each piece of Product B requires 10 work days, 5 kilowatt hours of electricity, and 4 tons of coals. The profit from each piece of Product A are given as in Table 1 and the profit from each piece of Product B is 1200$. This company has 300 movable laborers, 360 tons of daily consumable coals and 250 kilowatt hours daily electricity supplies.

Table 1 Profit from each piece of Product A during 2015 to 2019

Year	**2015**	**2016**	**2017**	**2018**	2019
profit from each piece of Product A ($)	500	560	640	750	870

For the year of 2020, how should the company arrange its daily production of Products A and B to maximize its profits?

Solution: Assume that the production of Products A and B are x_1 and x_2, respectively. Then, we have the following linear programming problem.

$$
\begin{aligned}
&\max Z = c_1 x_1 + 1200 x_2 \\
&s.t. 4x_1 + 5x_2 \leq 250 \\
&9x_1 + 4x_2 \leq 360, \\
&3x_1 + 10x_2 \leq 300, \\
&x_1, x_2 \geq 0.
\end{aligned}
$$

From Table 1, we obtain a sequence of restrained variables c_1 as follows, $c_1 = (c_1(\mathrm{i}))_{i=1}^{5} = (500, 560, 640, 750, 870)$. By using GM(1,1) model solved predicted value and we obtain $\hat{c}_1(6) \approx 1005$. Also, by using the $GM^{\frac{p}{q}}(1,1)$ model if $\frac{p}{q} = 0.1$ then we obtain $\hat{c}_1(6) \approx 1007$. Therefore, for fractional predicted model value, corresponding linear programming model for year of 2020 is as follows:

$$
\begin{aligned}
&\max Z = 1007 x_1 + 1200 x_2 \\
&s.t.\ 4x_1 + 5x_2 \leq 250 \\
&\quad 9x_1 + 4x_2 \leq 360, \\
&\quad 3x_1 + 10x_2 \leq 300, \\
&\quad x_1, x_2 \geq 0.
\end{aligned}
$$

This problem can be solved by introducing slack variables to convert to the standard form and then applying the simplex algorithm. We can find the optimal solutions as follows: $x_1 = 30.77$, $x_2 = 20.77$ and maximum daily profit obtain $Z = 55907.69$.

6 Conclusion

In optimization issues, we may have to predict prices due to uncertainties in data. If the number of data is low or we have incomplete information, the use of grey prediction would be very suitable. By using the gray forecast, the price of the next year will be achieved and the corresponding linear programming problem solved. The predictions were made using the GM(1,1) and fractional methods, and due to the closeness of the values, the problem was solved by using the obtained value obtain the fractional method.

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