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# Common set of weights: a double frontier DEA approach

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**Abstract** Data Envelopment Analysis (DEA) is a non-parametric method for efficiency measurement. In the most common DEA models the method selects the most favorable weight set for all units in order to maximize their efficiency scores. The so called optimistic assessment determines the best efficiency score. To make the performance of DMUs more actionable, the evaluation can be addressed from pessimistic perspective. Under the optimistic and pessimistic points of view, the performance of a unit is assessed with two different evaluation methods. As a result, a different set of weights is achieved for each unit. Hence, to have a more realistic results and better discrimination among DMUs, a more applicable method of a common set of weights (CSW) is suggested. The contribution of the paper is three folded. (1) The proposed approach develops the weight restriction approach, taking into account both optimistic and pessimistic points of view, simultaneously. (2) The proposed weight restriction method considering double frontier generates a positive and a dissimilar set of weights. (3) With the achieved common set of weights the efficiency scores are calculated then the units are ranked. To highlight the details of the proposed method, a real world data application consists of real case study confirm that the presented procedure results in a more realistic and the comprehensive assessment. It also shows the superiority of the proposed method considering double frontier.

**Keyword:** Data Envelopment Analysis, Common Set of Weights, Weight Restriction, Optimistic and Pessimistic Efficiencies, Double Frontier, Weight Dissimilarity.

## 1 Introduction

Data envelopment analysis (DEA) is concerned with a comparative assessment of the efficiency of decision making units (DMUs). In classical DEA models, the efficiency of a DMU is obtained by maximizing the ratio of the weighted sum of its outputs to the weighted sum of its inputs, subject to the condition that this ratio does not exceed one for any DMU. Since the pioneering work of Charnes et al. [1] and Banker et al. [2] the non-parametric mathematical programming DEA has demonstrated as an effective technique for measuring

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the relative efficiency of a set of homogeneous DMUs in productivity and efficiency analysis. Specifically, the flexibility of standard DEA models in choosing a set of weights for inputs and outputs, often causes more than one DMU being evaluated as efficient. What's more, leading them being unable to be fully discriminated. One of the possible ways for solving this problem settles in the specification of a common set of weights (CWS). Many researchers have proposed different approaches to achieve a common set of weights. For example, refer to Pourhabib et al. [3], Ramon et al. [4], Roll et al. [5], Wu et al. [6], Eyni et al. [7] and some other researchers. In before mentioned papers, the proposed method can be known as a method for analysis the best relative efficiency or optimistic efficiency. In their proposed model, a DMU is specified as DEA efficient or optimistic efficient if its best relative efficiency equals one; otherwise, it is called DEA-non-efficient or optimistic non-efficient. Having emphasis on the non-performing units, the performance of units can be evaluated from the pessimistic point of view. The worst relative efficiency or pessimistic point of view assigns the most unfavorable weights to each unit. If the optimal value of the model is equal to unity, that DMU is called as DEA-inefficient or pessimistic inefficient; otherwise, it is said to be DEA-pessimistic non efficient. In order to have a general scenario about the performance of a DMU, applying both points of view, optimistic and pessimistic is practically more useful. To over hatching the benefits of both perspectives in practice, Azizi [8] presented a bounded model for obtaining an interval efficiency using the concept of optimistic and pessimistic efficiencies. The author also highlighted the shortcoming of Entani's model, namely, Entani's model (2002) does not take all input and outputs in the evaluation, and so, it is not able to identify an adequate bound for interval efficiencies. Azizi et al. [8] pointed out to the drawback of existing model for evaluating interval efficiency and a proposed pair of revised models that make it possible to perform a DEA efficiency analysis based on the new interval efficiency models. Salahi et al. [9] suggested an equivalent formulation of the robust envelopment CCR model in the presence of input and output uncertainty. What's more, the authors proposed a linear programming for deriving a common set of weights (CSW) under uncertainty. Arabmaldar et al. [10] proposed an approach for handling uncertainty in presence of interval data. A key advantage of this approach is focusing on the worst performing frontier with non-discretionary factors. Using overall performance measures, Jahed et al. [11] proposed an overall performance measures for evaluating DMUs developing the fuzzy DEA theory and methodology. The authors proposed a fuzzy DEA models that evaluate a DMU from the pessimistic perspective in a fuzzy context. Finally, using the double frontier analysis approach, a measure for evaluating the performance is obtained. Tapia et al. [12] focused on the measuring efficiency problem as a statistical problem. The authors proposed two confidence interval methodologies. One is inspired in the optimistic/pessimistic point of view of DEA models and the other in the use of bootstrap replications from the sample of customers in each DMU. Reza Kiani Mavi et al. [13] employing the concept of ideal point derived a common set of weight for the Malmquist productivity index. The authors proposed a novel common set of weights model for double frontier DEA in presence of undesirable output and applied the results in the freight transportation in Iran. Amirteimoori et al. [14] applies a different angle of the double frontier concept and proposed a linear model without the need for additional changes in variables and use the same set of constraints to measure the efficiency of DMUs with fuzzy inputs and output in two-stage structures. Fathi and Farzipoor [15] assess the sustainability of supply chains contributing the knowledge of double frontier network DEA and common set of weights (CSW) in presence of fuzzy data set. The proposed model takes into account different confidence levels in two periods and can fully rank DMUs. In other research, Farzipoor Saen et al.[16] proposed a Malmquist

productivity index (MPI) based on network data envelopment analysis (NDEA) model in the presence of integer data, undesirable outputs, and non-discretionary inputs. In recent studies, Kutty et al. [17] addressed the concept of smart sustainable cities proposing a novel Double-Frontier Slack Based Measure Data Envelopment Analysis model in presence of undesirable factors. With six dimensions of sustainable development and in terms of optimistic and pessimistic viewpoints, the achieved interval efficiency is used to determine the most efficient smart city in Europe. Yu Sun et al. [18] presented approach comprehensively measures higher education system performance in terms of optimistic and pessimistic aspects. The results showed that the presented model has more ranking discrimination power than the traditional optimistic and pessimistic models. Salahi et al. [19] employed norm-1 and Bertsimas and Sim approach to achieve the common set of weights under DEA frameworks. The advantages of the proposed method are confirmed applying a real case study.
Optimistic and pessimistic efficiencies measure two extremes of each DMU performance. To determine the overall performance of each DMU, both perspectives should be considered simultaneously. An approach that evaluates the performance of each DMU for both optimistic and pessimistic efficiencies is called double frontier analysis approach. However, sometimes the researchers have made some contributions to deal with the common set of weights (CSW) employing double frontier analysis. Since, applying one of the efficiencies suffers from bias. In this paper, we aim to search one common set of weights employing the interval efficiency of each DMU and then rank the DMUs with these obtained interval efficiency scores. The proposed weight restriction approach generates positive weights and, at the same time, prevents weights dissimilarity when an interval efficiency is taken into account. The rest of the paper has the following order. The next section will present the basic DEA method for measuring interval efficiencies and weight restriction approach in DEA literature. In the section to follow a common set of weights (CSW) is found by employing an interval efficiency along with the weight restriction approach. Numerical examples are discussed in section4, and conclusions are offered in section section5.

## 2 Preliminaries

Since the performances of DMUs can be measured from both optimistic and pessimistic views, two efficiencies are obtained for each DMU: optimistic and pessimistic efficiency. Consider a set of DMU indexed by $J$. For all $j \in J = \{1,\dots,n\}$, $DMU_j$ uses input $x_{ij}(i=1,\dots,m)$ to produce $y_{rj}(r=1,\dots,s)$. Also, for each $j \in J$, the input and output value of $DMU_j$ are known and positive. The following multiplier form of CCR presented by Charnes et al. [1] measures the best relative efficiency of $DMU_j$:

$$
\begin{aligned}
\max \quad & \theta_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\
& \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1 \quad , j = 1,\dots,n \\
& u_r, v_i \ge \varepsilon \ , r = 1,\dots,s, i = 1,\dots,m
\end{aligned} \tag{1}
$$



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In the above model, $u_r(r=1,...,s)$ and $v_i(i=1,...,m)$ denote the weight value for $r-th$ output and $i-th$ input respectively, and $\varepsilon$ is a non-Archimedean infinitesimal number. Employing Charnes and Cooper [20] transformation, the above model is converted to linear programming model as follows:

$$\begin{aligned} \max\ \ & \theta_o = \sum_{r=1}^{s} u_r y_{ro} \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \quad , \ j=1,...,n \\ & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & u_r, v_i \ge \varepsilon \ , \ r=1,...,s \ , \ i=1,...,m \end{aligned} \tag{2}$$

The above linear model (2) measures the best relative efficiency of DMUs in the output-oriented mode. If the optimal value of the objective function in model (2) is one, $\theta_o^* = 1$ $DMU_o$ is said to be DEA-efficient or optimistic efficient; otherwise, it is DEA-non-efficient or optimistic non-efficient. From the pessimistic view, the worst efficiency score is evaluated relative to DMUs on the worst performing frontier. The following model is expressed as a pessimistic DEA model:

$$\begin{aligned} \min\ \ & \varphi_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\ & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \ge 1 \quad , \ j=1,...,n, \\ & u_r, v_i \ge \varepsilon \quad r=1,...,s, i=1,...,m \end{aligned} \tag{3}$$

Rearranging the model (3) by Charnes and Cooper [20] transformation, the problem (3) can be converted into a linear program:

$$\begin{aligned} \min\ \ & \varphi_o = \sum_{r=1}^{s} u_r y_{ro} \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \ge 0 \quad , \ j=1,...,n \\ & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & u_r, v_i \ge \varepsilon \ , r=1,...,s, i=1,...,m \end{aligned} \tag{4}$$



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The model (4) identifies the worst performing unit by assigning the most unfavorable weights to each DMU in the unfavorable scenario. If in optimality, $\varphi_o^* = 1$ $DMU_o$ is said to be DEA-inefficient or pessimistic inefficient.

## 3 Research Findings

Theoretically, the best and the worst relative efficiencies should form an interval. For this purpose, the pessimistic efficiency should be adjusted. Assume that $\alpha (0 < \alpha \le 1)$ is the adjustment factor. The adjusted interval efficiency can be written as $[\alpha_j \varphi_j^*, \theta_j^*] (j = 1,...,n)$ so that the condition $\alpha_j \varphi_j^* \le \theta_j^*, (j = 1,...,n)$ holds for all intervals $[\alpha_j \varphi_j^*, \theta_j^*] (j = 1,...,n)$. Pinning with this parameter $\alpha_j (j = 1,...,n)$, in order to search a positive lower bound for a common set of weights among all feasible multipliers, a joint weight restriction approach [3] is applied. The joint weight restriction approach proposed by Pourhabib et al. [3] allows selecting common weights through conjointly restricting the input and output weights with a common bound. Again, suppose there are $n$ units, and each unit uses input $x_{ij} (i = 1,...,m)$ to produce $y_{rj} (r = 1,...,s)$. Also, for each $j = 1,...,n$. The model proposed by Pourhabib et al. [3] has the following format:

$$\begin{aligned} \min \quad & \frac{\sum_{j=1}^{n} d_j}{\alpha} \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + d_j = 0 \quad , \; j = 1,...,n, \\ & \alpha \le v_i \le 1, \quad i = 1,...,m \\ & \alpha \le u_r \le 1, \quad r = 1,...,s \end{aligned} \tag{5}$$

In the model (5), the variable $d_j (j = 1,...,n)$ is denoted as deviation variable or slack variable for each unit and $\alpha$ shows lower bound for both input and output weight. Moreover, all weights do not exceed the upper bound (which is unity). The objective function minimizes the summation of deviation variable and maximizes the lower bound.

Equipped with this approach, in order to have a common set of weight considering adjusted interval efficiency, namely, $[\alpha_j \varphi_j^*, \theta_j^*] (j = 1,...,n)$ the following model can be structured:

$$
\begin{aligned}
& Min \quad \eta = \frac{\sum_{j=1}^{n} s_{1j} + \sum_{j=1}^{n} s_{2j}}{\delta} \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + s_{1j} = 0 \quad , \; j=1,\dots,n, \\
& \sum_{r=1}^{s} u_r y_{rj} - (\alpha_j) \sum_{i=1}^{m} v_i x_{ij} - s_{2j} = 0 \\
& \alpha_j . \varphi_j^* \le \theta_j^* \quad j=1,\dots,n \\
& \delta \le v_i \le 1, \quad i=1,\dots,m \\
& \delta \le u_r \le 1, \quad r=1,\dots,s
\end{aligned}
\tag{6}
$$

It is clear that the above model (6) is nonlinear. The objective function minimizes the summation of deviation variable and maximizes the lower bound of weights. As the third constraint claims $\theta_j^*$ and $\varphi_j^*$ are the upper and lower bound of interval efficiency, respectively. They actually form an interval efficiency $[\varphi_j^*, \theta_j^*](j=1,\dots,n)$. Theoretically the lower bound of the interval should be adjusted, the variable $\alpha_j(j=1,\dots,n)$ holds the condition that $\alpha_j \varphi_j^* \le \theta_j^*, (j=1,\dots,n)$. Therefore, employing the best and the worst relative efficiencies, the proposed weight restriction approach generates positive weights and prevents weight dissimilarity. In a nutshell, model (6) finds a common set of weights using the best and worst efficiency scores. The positive and non-zero weights are also applied for adequate ranking of DMUs. The following theorem proves that the proposed weight restriction approach is feasible and generates positive weights.

Theorem1: Model (6) is always feasible and generates positive weights in optimality.

Proof: refer to Pourhabib et al.[3].

## 4 Illustrative Application

The applicability of the proposed approach is illustrated by two real data set. In the first example, seventeen forest district from Kao and Hung [21] are given. Four inputs, including Budget in US dollars ($I1$), initial stocking in cubic meters ($I2$), labor in number of employees ($I3$) and land in hectares ($I4$) are used to produce three outputs, namely, main product in cubic meters ($O1$), soil conversation in cubic meters ($O2$) and recreation in number of visits ($O3$). Table1 shows Data set.

**Table 1** Data Set of seventeen forest districts

| DMU | $I1$ | $I2$ | $I3$ | $I4$ | $O1$ | $O2$ | $O3$ |
|---|---|---|---|---|---|---|---|
| DMU1 | 51.62 | 11.23 | 49.22 | 33.52 | 40.49 | 14.89 | 3166.71 |
| DMU2 | 85.78 | 123.98 | 55.13 | 108.46 | 43.51 | 173.93 | 6.45 |
| DMU3 | 66.65 | 104.18 | 257.09 | 13.65 | 139.74 | 115.96 | 0 |
| DMU4 | 27.87 | 107.6 | 14 | 146.43 | 25.47 | 131.79 | 0 |
| DMU5 | 51.28 | 117.51 | 32.07 | 84.5 | 46.2 | 144.99 | 0 |
| DMU6 | 36.05 | 193.32 | 59.52 | 8.23 | 46.88 | 190.99 | 822.29 |



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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| DMU7 | 25.83 | 105.8 | 9.51 | 227.2 | 19.4 | 120.09 | 0 |
| DMU8 | 123.02 | 82.44 | 87.35 | 98.8 | 43.33 | 125.84 | 404.69 |
| DMU9 | 61.95 | 99.77 | 33 | 86.37 | 45.43 | 79.6 | 1252.6 |
| DMU10 | 80.33 | 104.65 | 53.3 | 79.06 | 27.28 | 132.49 | 42.76 |
| DMU11 | 205.92 | 183.49 | 144.16 | 59.66 | 14.09 | 196.29 | 16.15 |
| DMU12 | 82.09 | 104.94 | 46.51 | 127.25 | 44.87 | 108.53 | 0 |
| DMU13 | 202.21 | 187.74 | 149.39 | 93.65 | 44.97 | 184.77 | 0 |
| DMU14 | 67.55 | 82.83 | 44.37 | 60.85 | 26.04 | 85 | 23.95 |
| DMU15 | 72.6 | 132.73 | 44.46 | 173.48 | 5.55 | 135.65 | 24.13 |
| DMU16 | 84.83 | 104.28 | 159.12 | 171.11 | 11.53 | 110.22 | 49.09 |
| DMU17 | 71.77 | 88.16 | 69.19 | 123.14 | 44.83 | 74.54 | 6.14 |

Models (2) and (4) are performed on the data set of Table1. The results of optimistic and pessimistic efficiencies are recorded in Table 2.

**Table 2** The results of models (2) and (4)

| DMU | $\theta_o^*$ | $\varphi_o^*$ |
|---|---|---|
| DMU1 | 1 | 1 |
| DMU2 | 1 | 1.96 |
| DMU3 | 1 | 1 |
| DMU4 | 1 | 1.14 |
| DMU5 | 0.95 | 1.24 |
| DMU6 | 1 | 1.07 |
| DMU7 | 1 | 1 |
| DMU8 | 0.78 | 1.10 |
| DMU9 | 0.90 | 1 |
| DMU10 | 0.65 | 1.30 |
| DMU11 | 0.74 | 1 |
| DMU12 | 0.47 | 1 |
| DMU13 | 0.52 | 1 |
| DMU14 | 0.59 | 1.09 |
| DMU15 | 0.53 | 1 |
| DMU16 | 0.48 | 1 |
| DMU17 | 0.42 | 1 |
| Average | 0.76 | 1.11 |
| variance | 0.05 | 0.05 |

Equipped with these efficiencies, model (6) is performed on the data set of Table1. The common set of weights generated by model (6) is presented in Table3.

**Table 3** Common set of weights generated by model (6)

| | INPUT | | | | OUTPUT | | |
|---|---|---|---|---|---|---|---|
| | $I1$ | $I2$ | $I3$ | $I4$ | $O1$ | $O2$ | $O3$ |
| Common weights | 0.01 | 0.01 | 0.03 | 1 | 0.18 | 0.01 | 1 |

Table4 represents the efficiency score of these seventeen forest districts employing the common set of weights recorded in Table 3.

**Table 4** The result of Common weights for DMUs

| DMU | Efficiency with common weights | Rank | $\theta_o^*$ Optimistic Efficiency Model(2) | Rank |
|---|---|---|---|---|
| DMU1 | 0.96327 | 1 | 1 | 1 |
| DMU2 | 0.096286 | 8 | 1 | 1 |
| DMU3 | 0.263128 | 3 | 1 | 1 |
| DMU4 | 0.059025 | 12 | 1 | 1 |
| DMU5 | 0.097659 | 7 | 0.95 | 2 |
| DMU6 | 0.771506 | 2 | 1 | 1 |
| DMU7 | 0.046929 | 15 | 1 | 1 |
| DMU8 | 0.129688 | 5 | 0.78 | 4 |
| DMU9 | 0.230512 | 4 | 0.90 | 3 |
| DMU10 | 0.067535 | 11 | 0.65 | 6 |
| DMU11 | 0.04737 | 14 | 0.74 | 5 |
| DMU12 | 0.091619 | 9 | 0.47 | 11 |
| DMU13 | 0.099423 | 6 | 0.52 | 9 |
| DMU14 | 0.059133 | 13 | 0.59 | 7 |
| DMU15 | 0.024919 | 16 | 0.53 | 8 |
| DMU16 | 0.034537 | 17 | 0.48 | 10 |
| DMU17 | 0.088632 | 10 | 0.42 | 12 |
| Average | 0.19 | --- | 0.76 | |
| Variance | 0.07 | --- | 0.05 | |



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Regarding to Table4, efficiency scores calculated by the obtained common set of weights, model (6), are recorded in the second column of Table4. It can be seen that the proposed weight restriction approach along with the adjusted interval efficiency has more discrimination on DMUs. From the statistical point of view, reported in the last row of Table4, the proposed approach attains the least value, 0.19. While, the average score of efficiency is 0.76 in optimistic evaluation. The results imply that the variance value in the proposed common set of weights is about 0.07 which is larger than the optimistic evaluation. For more comparison, the results are compared with the robust DEA model presented by Salahi et al.[9]. Their model distinguishes on interval uncertainties on input and output data. That is to say, $x_{ij} \in \left[\underline{x}_{ij}, \overline{x}_{ij}\right]$ and $y_{rj} \in \left[\underline{y}_{rj}, \overline{y}_{rj}\right]$ for $i=1,..,m$, $r=1,\dots,s$ and $j \in J=\{1,\dots,n\}$. Where $\underline{x}_{ij}, \overline{x}_{ij}, \underline{y}_{rj}, \overline{y}_{rj}$ are known. The proposed robust model has the following format:

$$
\begin{aligned}
&\theta_j^* = Max \ \sum_{r=1}^{s} u_r \overline{y}_{ro} \\
&s.t. \\
&\qquad \sum_{r=1}^{s} u_r \underline{y}_{rj} - \sum_{i=1}^{m} v_i \overline{x}_{ij} \le 0 \quad , j=1,\dots,n \\
&\qquad \sum_{i=1}^{m} v_i \underline{x}_{io} \le 1 \ , \\
&\qquad \sum_{i=1}^{m} v_i \overline{x}_{io} \ge 1, \\
&\qquad u_r, v_i \ge \varepsilon \quad , i=1,\dots,m, r=1,\dots s.
\end{aligned}
\tag{7}
$$

Model (7) is applied for robust efficiency evaluation. Model (7) is called as optimistic counterpart of the multiplier format of the standard CCR model. Then employing the optimal solution of model (7), $\theta_j^*$, the common set of weights are obtained using the following robust model:

$$\begin{aligned} & Min \ \sum_{j=1}^{n}(\theta_j^* \sum_{i=1}^{m} v_i \bar{x}_{ij} - \sum_{r=1}^{s} u_r \underline{y}_{rj}) \\ & s.t. \\ & \quad \sum_{r=1}^{s} u_r \bar{y}_{rj} - \sum_{i=1}^{m} v_i \underline{x}_{ij} \le 0 \quad , j = 1,...,n, \\ & \quad u_r, v_i \ge \varepsilon \quad , i = 1,...,m, r = 1,...s. \end{aligned} \tag{8}$$

The nonlinear model (8) employs the robust efficiency score of model (7), namely $\theta_j^*$, to compute the common set of weights under interval uncertainties. Finally, applying the optimal common weights of model (8), $(u^*, v^*)$, the ratio $\theta_j^{CWR} = \frac{\sum_{r=1}^{s} u_r^* y_{rj}}{\sum_{i=1}^{m} v_i^* x_{ij}}$ computes the efficiency score of under evaluated units under the interval uncertainties. The common set of weights generated for seventeen data set by model (8) is presented in Table 5.

**Table 5** Common set of weights generated by Salahi's et al.model (8)

| | INPUT | | | | OUTPUT | | |
|---|---|---|---|---|---|---|---|
| | $I1$ | $I2$ | $I3$ | $I4$ | $O1$ | $O2$ | $O3$ |
| Common weights | 0.0001 | 0.0001 | 0.0065 | 0.0009 | 0.0001 | 0.0017 | 0.0001 |

The results of robust efficiency, model (8), for seventeen data set in Table1, are recorded in Table6.

**Table 6** The result of efficiency score of robust DEA model (8)

| DMU | Robust Efficiency with common weights | Rank |
|---|---|---|
| DMU1 | 0.9734 | 2 |
| DMU2 | 0.6260 | 11 |
| DMU3 | 0.1228 | 17 |
| DMU4 | 0.9595 | 3 |
| DMU5 | 0.8282 | 5 |
| DMU6 | 0.9773 | 1 |
| DMU7 | 0.7409 | 6 |
| DMU8 | 0.3795 | 15 |



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| | | |
|---|---|---|
| DMU9 | 0.8599 | 4 |
| DMU10 | 0.5284 | 14 |
| DMU11 | 0.3236 | 16 |
| DMU12 | 0.6817 | 9 |
| DMU13 | 0.6139 | 12 |
| DMU14 | 0.6978 | 8 |
| DMU15 | 0.7291 | 7 |
| DMU16 | 0.6574 | 10 |
| DMU17 | 0.5850 | 13 |
| Average | 0.6637 | ----- |
| Variance | 0.0547 | ------- |

As Salahi et al.[9] declared that for employing model (7) and model(8), the data set is assumed to settle in intervals as $x_{ij} \in \left[\underline{x}_{ij}, \bar{x}_{ij}\right] = \left[x_{ij} - \delta, x_{ij} + \delta\right]$ and $y_{rj} \in \left[\underline{y}_{rj}, \bar{y}_{rj}\right] = \left[y_{rj} - \delta, y_{rj} + \delta\right]$ with $\delta = 1$ and $\varepsilon = 0.0001$. Looking closely to the results, the proposed model (6) has a more reliable ranking of DMUs. It is worth to note that, model (8) catches the defined $\varepsilon = 0.0001$ as a common weight for some inputs and outputs, whereas the proposed model (6) gives different values as common eights. This point indicates that proposed weight restriction approach, model (6), not only leads to strictly positive weights but also prevents dissimilar weights. Thus, the main advantages of the proposed method are applicable for nominal data and can do a complete ranking of DMUs. Statistically speaking, the average of robust DEA model, model (8), 0.6637, is larger than its classical counterpart in model (6), 0.19.

The second example is taken from Salahi et al. [9] and consists of twenty-seven Iranian gas companies in 2008. The data set consumes two inputs, $(I1)$ and $(I2)$ to generate two outputs $(O1)$ and $(O2)$. Table7 records data set.

**Table 7** Data Set of twenty-seven gas companies in 2008

| DMU | $I1$ | $I2$ | $O1$ | $O2$ |
|---|---|---|---|---|
| DMU1 | 3167.7 | 148 | 976.3 | 209.889 |
| DMU2 | 5177.8 | 197 | 2820.3 | 420.07 |
| DMU3 | 10664.8 | 355 | 6645.4 | 831.751 |
| DMU4 | 978.8 | 41 | 5477.2 | 7.911 |
| DMU5 | 3411.5 | 150 | 978.1 | 165.384 |
| DMU6 | 16545.2 | 754 | 14.861 | 1069.452 |
| DMU7 | 11088.9 | 512 | 7913 | 630.757 |
| DMU8 | 2614.9 | 141 | 3874.1 | 195.17 |
| DMU9 | 2242.2 | 123 | 1821.9 | 227.171 |
| DMU10 | 9398.4 | 384 | 4529 | 465.329 |
| DMU11 | 4654.5 | 186 | 1222.2 | 267.941 |
| DMU12 | 5424.7 | 281 | 2526.2 | 317.115 |
| DMU13 | 497.6 | 80 | 29.6 | 11.345 |
| DMU14 | 6171.4 | 214 | 7119 | 269.039 |
| DMU15 | 3515.1 | 143 | 2315 | 273.419 |
| DMU16 | 11111.8 | 448 | 8233.7 | 1118.628 |
| DMU17 | 2149.9 | 88 | 2269.2 | 120.477 |
| DMU18 | 7508.7 | 543 | 11366.99 | 434.583 |
| DMU19 | 1791.4 | 117 | 437.7 | 72.779 |
| DMU20 | 2916.5 | 127 | 1732.2 | 193.985 |
| DMU21 | 2645.2 | 156 | 1328.6 | 240.223 |



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| | | | | |
|---|---|---|---|---|
| DMU22 | 4614.2 | 204 | 3970 | 290.994 |
| DMU23 | 11445.1 | 783 | 5842 | 670.312 |
| DMU24 | 2784.4 | 139 | 1054 | 145.934 |
| DMU25 | 20981.1 | 1130 | 24353.4 | 2091.476 |
| DMU26 | 3993.5 | 142 | 2176.4 | 206.249 |
| DMU27 | 1992.7 | 117 | 888 | 138.526 |

Running models (2), (4), (6) and (8) on the data set of Table7, the common set of weights achieved from model (6) and model (8) is presented in Table 8.

**Table 8** Common set of weights generated by Model (6) and Model (8)

| | INPUT | | OUTPUT | |
|---|---|---|---|---|
| | $I1$ | $I2$ | $O1$ | $O2$ |
| Common weights of Model(8) | 0.0005 | 0.0027 | 0.0001 | 0.0053 |
| Common weights of Model(6) | 0.12 | 0.02 | 0.02 | 1 |

Referring to Table8, the achieved weights of model (6) are greater than their counterpart in model (8). The weights in model (8) are almost close to zero, whilst the model (6) prevents zero weights and seems that model (6) have more reliable weight values for evaluation. Equipped with the common set of weights of Table8, the efficiency score and ranking of twenty seven DMUs are depicted in Table9.

**Table 9** The result of efficiency score of Models (6), (2) and model (8)

| DMU | Efficiency with common weights Model(6) | Rank | $\theta_o^*$ Optimistic Efficiency Model(2) | Rank | Robust Efficiency with common weights Model(8) | Rank |
|---|---|---|---|---|---|---|
| DMU1 | 0.598863 | 15 | 0.6563 | 14 | 0.6104 | 16 |
| DMU2 | 0.762025 | 7 | 0.8540 | 4 | 0.8043 | 8 |
| DMU3 | 0.749613 | 9 | 0.9486 | 2 | 0.8070 | 6 |
| DMU4 | 0.993059 | 2 | 1 | 1 | 0.9857 | 3 |
| DMU5 | 0.448484 | 25 | 0.4808 | 22 | 0.4410 | 25 |
| DMU6 | 0.53474 | 20 | 0.6957 | 8 | 0.6943 | 10 |
| DMU7 | 0.58842 | 16 | 0.5984 | 16 | 0.5974 | 17 |
| DMU8 | 0.861166 | 5 | 0.8510 | 5 | 0.8433 | 5 |
| DMU9 | 0.97085 | 3 | 1 | 1 | 0.9547 | 4 |
| DMU10 | 0.489577 | 24 | 0.5102 | 21 | 0.4819 | 24 |
| DMU11 | 0.520017 | 21 | 0.5769 | 19 | 0.5112 | 22 |
| DMU12 | 0.559927 | 18 | 0.5780 | 18 | 0.5509 | 21 |
| DMU13 | 0.194693 | 27 | 0.2250 | 24 | 0.1913 | 27 |
| DMU14 | 0.552353 | 19 | 0.6711 | 11 | 0.5848 | 18 |
| DMU15 | 0.752861 | 8 | 0.7858 | 6 | 0.7846 | 9 |
| DMU16 | 0.955993 | 4 | 1 | 1 | 0.9987 | 2 |



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| | | | | | | |
|---|---|---|---|---|---|---|
| DMU17 | 0.638546 | 14 | 0.6624 | 13 | 0.6601 | 13 |
| DMU18 | 0.725869 | 10 | 0.7209 | 7 | 0.6599 | 14 |
| DMU19 | 0.375196 | 26 | 0.4010 | 23 | 0.3690 | 26 |
| DMU20 | 0.648556 | 12 | 0.6683 | 12 | 0.6675 | 12 |
| DMU21 | 0.832319 | 6 | 0.8963 | 3 | 0.8069 | 7 |
| DMU22 | 0.664046 | 11 | 0.6807 | 10 | 0.6792 | 11 |
| DMU23 | 0.566675 | 17 | 0.5799 | 17 | 0.5576 | 19 |
| DMU24 | 0.495726 | 23 | 0.5184 | 20 | 0.4867 | 23 |
| DMU25 | 1.005000 | 1 | 1 | 1 | 0.9994 | 1 |
| DMU26 | 0.518145 | 22 | 0.6141 | 15 | 0.5512 | 20 |
| DMU27 | 0.647243 | 13 | 0.6861 | 9 | 0.6276 | 15 |
| Average | 0.654337 | | 0.698515 | | 0.663207 | |
| Variance | 0.040621 | | 0.037577 | | 0.037808 | |

According to Table 9, as the efficiency scores of models admit, DMU#25 has the first score in all three models. Notably, model (6) and model (8) ranks DMU#13 as the last unit in the series, whilst, the optimistic efficiency model (2) assigns the 24th ranking location for this unit. From the statistical point view, it can be seen that the proposed weight restriction approach has the lowest average quantity compared with the other two model (2) and model (8). It is worth no note that, the variance quantity is almost close together in evaluation with three models. In a nutshell, the main advantage of the proposed weight restriction method is to find reliable common weight values to employ in efficiency evaluation and can do a complete ranking of DMUs.

## 5 Conclusion

Standard DEA models suffers flexibility in selecting inputs / output weights for evaluating the efficiency scores. On the other hand, the conventional forms of DEA models evaluate DMUs from the optimistic point of view. In order to obtain an overall assessment of the performance of each DMU, we need to integrate different performance measures. That is to say, the double frontier evaluation dare to be employed, i.e., the performance of a unit consists of both optimistic and pessimistic points of view. Equipped with both evaluations, this paper employs a joint weight restriction approach to generate a common set of weights for all DMUs. A key advantage of this approach is focusing on both evaluation to identify positive and dissimilar weights for inputs and outputs. The practical application of this methodology for evaluating two real practical case studies illustrated the strength of developed weight restriction approach in generating positive and dissimilar weights.

## References

1. Charnes, A., Cooper, W.W., Rhodes, E., (1978), Measuring the efficiency of decision making units, European Journal of Operational Research, 2(6), 429-444.
2. Banker, R.D., Charnes, A., Cooper, W.W, (1984), Some Models for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis, Management Science, 30(9).
3. Pourhabib, A., Kordrostami, S., Amirteimoori, A., Kazemi Matin, R., (2018), Data envelopment analysis with common weights: the weight restriction approach, Mathematical Science, 12,197-203.
4. Ramon, N., Ruiz,J., Sirvent, J.,(2012), Common set of weights as summaries of DEA profiles of weight :An application to the ranking of professional tennis players, Expert Systems with Applications, 39, 4882-48.
5. Roll, Y., Golany, B., (1993), Alternative methods for treating factor weights in DEA, Omega, 21, 99-109.



[ DOI: 10.71885/ijorlu-2023-1-644 ]

6. Wu, J., Chu, J., Qingyuan, Z., Yongjun, L., Liang, L., (2017), Determining Common weights in data envelopment analysis based on satisfaction degree , Journal of Operation Research Society, 67, 1446-1458.
7. Eyni, M., Maghbouli, M., (2014), Deriving Common Set of Weights in the Presence of the Undesirable Inputs: A DEA based Approach, International Journal of Data Envelopment Analysis, 2(4), 497-508.
8. Azizi, H., Jahed, Rasul, (2011), Improved data envelopment analysis models for evaluating interval efficiencies of decision-making units, Computers and Industrial Engineering, 61,897-901.
9. Salahi, M., Torabi, N., Amiri, A., (2016), An optimistic robust optimization approach to common set of weights in DEA, Measurement, 93, 67-73.
10. Arabmaldar, A., Kwasi Mensah, E., Toloo, M, (2021), Robust worst-practice interval DEA with non-discretionary factors, Expert Systems with Applications, 182,115256.
11. Jahed, R., Amirteimoori, A., Azizi, H., (2015), Performance measurement of decision-making units under uncertainty conditions: An approach based on double frontier analysis, Measurement, DOI: 10.1016/j.measurement.2015.03.014.
12. Tapia, J.A., Salvador, B., Rodriguez, J.M., (2019),Data Envelopment Analysis with Estimated Output Data: Confidence Intervals Efficiency, Measurement, DOI:10.1016/j.measurement.2019.107364.
13. Kiani Mavi, R., Fathi, A., Farzipoor Saen, R.,Kiani Mavi, N,(2019), Eco-innovation in transportation industry: A double frontier common weights analysis with ideal point method for Malmquist productivity index, Resources, Conservation and Recycling,147,39-48.
14. Amirteimoori, A., Azizi, H., Kordrostami, S., (2020), Double Frontier Two-Stage Fuzzy Data Envelopment Analysis, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 28(1), 117-152.
15. Fathi, A., Farzipoor Saen, R, (2021), Assessing sustainability of supply chains by fuzzy Malmquist network data envelopment analysis: Incorporating double frontier and common set of weights, Applied Soft Computing, 113, Part A, 107923.
16. Farzipoor Saen, R., Karimi, B, Fathi, A, (2022), Assessing the sustainability of transport supply chains by double frontier network data envelopment analysis, Journal of Cleaner Production, 354,131771.
17. Kutty, A.A., Kucukvar, M., Abdella, G.M., Bulak, M.E., Onat, M.N, (2022), Sustainability Performance of European Smart Cities: A Novel DEA Approach with Double Frontiers, Sustainable Cities and Society, 81,103777.
18. Sun, Y., Wang, D., Yang, F., Ang, S, (2023), Efficiency evaluation of higher education systems in China: A double frontier parallel DEA model, Computers and Industrial Engineering, 176,108979.
19. Salahi, M., Toloo, M., Torabi, N., (2020), A new robust optimization approach to common weights formulation in DEA, Journal of the Operational Research Society, 72(6), 1390-1402.
20. Charnes, A., Cooper, W.W, (1962),Programming with linear fractional functions, New Research Logistic Quarterly, 9(3-4), 181-186.
21. Kao, C., Hung, H.T., (2005)Data envelopment analysis with common weights: the compromise solution approach, Journal of Operation Research Society, 56(10), 1196-1203.