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Development of returns to scale in two-stage network DEA via parametric analysis

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Abstract Network Data Envelopment Analysis (DEA) has attracted considerable attention in both methodological research and practical applications of performance evaluation. This paper investigates a fundamental class of network DEA models, namely the two-stage network framework. In conventional DEA, where the production process is viewed as a "black box," returns to scale (RTS) plays a critical role in guiding managerial decisions on whether to expand or contract operations. This study extends the traditional concept of RTS to a two-stage network by examining input variations from three perspectives: stage 1, stage 2, and the overall system. The proposed approach employs parametric analysis to capture how these variations affect the relationships among inputs, intermediate measures, and final outputs. To ensure practical applicability, the method can be implemented through existing linear programming formulations and remains computationally feasible even for larger-scale problems. In addition, we develop a linear programming model that supports central managers in coordinating resource allocation across different stages, thereby achieving system-wide improvements. A numerical example illustrates that RTS classifications at the system and sub-process levels may diverge, offering distinct insights into pathways for enhancing productivity.

Keyword: Data Envelopment Analysis, Returns to scale, Network.

1 Introduction

Data Envelopment Analysis (DEA) is a nonparametric method used to assess the relative efficiency of a set of homogeneous Decision Making Units (DMUs), each of which applies multiple inputs to generate various outputs. The DEA technique, introduced by Charnes, Cooper, and Rhodes [1], is based on mathematical programming. Returns to Scale (RTS) represent a crucial concept in analyzing the efficiency of DMUs in applied production analysis within organizations. RTS provides managers with valuable insights on whether expanding or contracting operations of a given DMU would be beneficial. It quantifies the marginal returns derived from an additional input in the production function, denoted as $y=f(x)$. To better understand RTS, consider a situation where the input increases from x to αx

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(with $\alpha>1$), resulting in a change in output from y to βy. Depending on the relationship between α and β, three cases can be identified:

- When $1<\alpha<\beta$: This indicates increasing returns to scale, meaning that a proportional increase in inputs leads to a higher percentage increase in outputs. This suggests economies of scale, and expanding operations would be beneficial.
- When $\alpha=\beta$: This signifies constant returns to scale, where a proportional increase in inputs results in an identical increase in outputs. The DMU operates at its optimal scale, and further changes may not significantly impact productivity.
- When $\alpha>\beta>1$: This implies decreasing returns to scale, where the proportional increase in inputs leads to a smaller increase in outputs. This suggests diseconomies of scale, and reducing operations may be necessary to maintain efficiency.

By evaluating the relationship between α and β, managers can gain valuable insights into the scale efficiency of the DMU and make more informed decisions regarding future operations.

In economics, RTS is sometimes defined through elasticity. Panzar and Willig [2] state that if the elasticity is greater than one, increasing returns to scale are present for the DMU. If elasticity equals one, the returns to scale are constant, and if less than one, it indicates decreasing returns to scale. Essentially, elasticity measures the proportional change in outputs relative to changes in inputs at a local level. Several perspectives on RTS calculation have emerged. One approach is based on the RTS definition in DEA models introduced by Banker [3]. Hadjicostas and Soteriou [4] developed a theoretical framework for analyzing one-sided elasticities in multi-output production systems, linking asymmetric output responses to technical efficiency and returns to scale, though their model remains purely theoretical and lacks empirical or network-based validation. Another method, proposed by Zarepisheh and Soleimani-Damaneh [5], focuses on calculating the proportional variations of outputs relative to inputs in disjoint intervals. The literature on DEA has extensively discussed the theory and application of RTS. Banker [3] highlights contradictions regarding RTS judgments based on the Most Productive Scale Size (MPSS). Banker and Thrall [6] present a method for estimating RTS using fractional DEA models, while Golany and Yu [7] introduce the concepts of right and left RTS. Tone [8] proposed a DEA model with weight restrictions, and Soleimani-Damaneh [9] simplified Tone's RTS definition. Khoveyni et al. [10] developed an approach to determine and measure right and left returns to scale for efficient DMUs. Krivonozhko et al. [11] presented a two-stage approach for measuring RTS in non-radial DEA models. Recently, Soleimani-Damaneh and Mostafaee [12] proposed additional classes to global RTS in FDH models. Although multiple studies have addressed RTS from various perspectives, many fail to account for the complexity of the production process, often treating it as a "black box" without considering intermediate products and subprocesses. In reality, production systems are intricate, and their internal structures must be taken into account when examining RTS. Hassanzadeh and Mostafaee [13] presented six scenarios related to link control, considering both cooperative and non-cooperative link control between the previous and next stages. Zhang et al. [14] studied RTS in a two-stage production process with a network structure. They analyzed how changes in the initial input affect the overall output in different scenarios. In the first scenario, the goal was to maximize the ratio of intermediate output to the initial input, with Stage 1 as the "leader" and Stage 2 as the "follower". The second scenario aimed at maximizing the sum of output-input ratios of both stages while

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considering their interrelationship. Upon reviewing the literature on Network Data Envelopment Analysis (NDEA), it is evident that RTS within this framework has not been sufficiently emphasized. This raises the important question of how to accurately assess RTS in a network structure, which is an exciting area for further research and empirical exploration in NDEA. The main objective of this research is to analyze the relationship between the proportional changes in outputs and inputs. This analysis demonstrates how calculating the rate of variation can assist managers in making better decisions. The approach introduced here determines this relationship through parametric analysis and perturbation in linear programming. Unlike existing literature, this method measures the rate of change in output relative to input during separate intervals, with the first interval's rate equaling the scale elasticity measure. The results obtained can significantly aid managers in deciding whether to expand or contract the DMU's operations. Moreover, the proposed parametric model is designed to perform efficiently even in large-scale systems, ensuring robustness and scalability for practical applications.

In recent years, numerous studies in Data Envelopment Analysis (DEA) with two-stage network structures have focused on decomposing efficiency and analyzing returns to scale (RTS). Sahoo et al. [15] proposed a framework for decomposing technical efficiency and scale elasticity, but their model was limited to small-scale examples and lacked full consideration of complex interdependencies. Subsequent studies have addressed these limitations in various ways. Sarparast et al. [16] investigated the sustainability of RTS classification in two-stage networks, although their approach faced challenges in computational complexity and generalizability to larger systems. Sueyoshi et al. [17] developed a time-dependent DEA model for evaluating operational efficiency and RTS in China's electricity sector, revealing significant spatial and temporal variations, but requiring substantial computational effort and specific time-series data. Amirteimoori et al. [18] proposed a fully fuzzy two-stage DEA model to evaluate RTS in Iran's airline sector, offering a novel approach under uncertainty but facing limitations due to model complexity and strict data requirements. Similarly, Amirteimoori et al. [19] presented a two-stage DEA model for the European forestry sector, providing useful insights into optimal scale size, but the model's high computational demands and context-specific applicability limit its generalizability.These studies highlight that despite valuable progress, there remains a need for DEA models capable of analyzing RTS in network structures with both methodological rigor and practical efficiency for larger-scale applications. The present research aims to address this gap by developing an integrated parametric model that:

1. Measures the rate of change in outputs relative to inputs in separate intervals.
2. Maintains computational efficiency for large-scale systems.
3. Provides robust guidance for managers to make informed decisions on scaling operations.

This approach extends existing literature by explicitly incorporating network structure considerations into RTS analysis, thereby enhancing the accuracy and practical applicability of DEA in complex production systems.

The computational framework of the proposed parametric model follows the linear programming and network flow theory discussed by Bazaraa, Jarvis, and Sherali [19], which ensures that the model remains mathematically consistent and computationally tractable even for larger-scale systems.

The remainder of the paper is organized as follows: Section 2 provides an overview of RTS in the BCC model and introduces the two-stage production process. Section 3 discusses the practical implications of understanding the relationship between proportional variations in

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outputs and inputs, including a motivating example. Section 4 presents key results and introduces an algorithm for determining this relationship. Section 5 includes a numerical example and proofs of the main results. Finally, Section 6 concludes the paper with some remarks.

2 Some basis concepts

Consider n DMUs, where DMU_j, $(j = 1,\ldots,n)$, produces s outputs: y_{rj} $(r = 1,\ldots,s)$, using m inputs: x_{ij} $(i = 1,\ldots,m)$. Define $x_j = (x_{1j}, x_{2j},\ldots,x_{mj})^T$ and $y_j = (y_{1j}, y_{2j},\ldots,y_{sj})^T$ as the input and output vectors of DMU_j. Also, let $X = [x_1, x_2,\ldots,x_n]$ and $Y = [y_1, y_2,\ldots,y_n]$ be the $m \times n$ and $s \times n$ matrices of inputs and outputs, respectively. We assume that the inputs and outputs are non-negative, and x_j and y_j are nonzero vectors for each j. The production possibility set T is represented as:

$$T = \{(x,y) \in \mathbb{R}_+^{m+s} | y \text{canbeproducedfrom} x\}.$$

Banker et al. [3] developed a production possibility set, denoted as T_{BCC}, by making certain assumptions related to variable returns to scale in the production technology.

$$T_{BCC} = \{(x,y) \in \mathbb{R}_+^{m+s} | X\lambda \leq x, Y\lambda \geq y, e\lambda = 1, \lambda \geq 0\}$$

where e and 0 are vectors with all components equal to one and zero, respectively. Regarding this PPS, they introduced an envelopment form BCC model to evaluate the performance of a specific DMU, denoted as $DMU_o(x_o, y_o)$, where $o \in \{1,2,\ldots,n\}$, as follows:

$$\theta_o^{BCC} = \min\{\theta | X\lambda \leq \theta x_o, Y\lambda \geq y_o, e\lambda = 1, \lambda \geq 0\}$$

or equivalently: $\theta_o^{BCC} = \min\{\theta | (\theta x_o, y_o) \in T_{BCC}\}.$

Let ∂T_{BCC} denote the boundary or frontier of T_{BCC}. The set of weak BCC-efficient points in input-orientation is represented by W_{BCC}^I, and the set of weak BCC-efficient points in output-orientation is represented by W_{BCC}^O. It should be noted that the "boundary" refers to the union of W_{BCC}^I and W_{BCC}^O. Recall that a production point $(x_o, y_o) \in T_{BCC}$ is considered as a weak BCC-efficient point in input-orientation if $\theta_o^{BCC} = 1$, and it is considered as a weak BCC-efficient point in output-orientation if $\varphi_o^{BCC} = 1$. Here, $\varphi_o^{BCC} = \max\{\varphi | (x_o, \varphi y_o) \in T_{BCC}\}$.

This passage focuses on Banker's concept of RTS (Returns to Scale), as defined in [3]. The definition is explained by considering a specific DMU (x_o, y_o) and evaluating it within the context of T_{BCC}. The definition applies for any value of $\beta > 0$.

$$\alpha(\beta) = \max\{\alpha | (\beta x_o, \alpha y_o) \in T_v\} \quad (1)$$

$$\gamma^+ = \lim_{\beta \to 1^+} \frac{\alpha(\beta) - 1}{\beta - 1}, \quad \gamma^- = \lim_{\beta \to 1^-} \frac{\alpha(\beta) - 1}{\beta - 1} \quad (2)$$

Definition 1. IRS (Increasing Returns to Scale) prevail at (x_o, y_o) if $\gamma^+ > 1$ and $\gamma^- > 1$; DRS (Decreasing Returns to Scale) prevail at (x_o, y_o) if $\gamma^+ < 1$ and $\gamma^- < 1$; otherwise, CRS (Constant Returns to Scale) prevail at (x_o, y_o).

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Hadjicostas et al. [14] examined the properties of Banker's limits, as defined earlier. One of the outcomes of their research is stated in the following lemma:

Lemma 1. If $\alpha(1) = 1$ and there exist $B \in [0,1)$ and $A \geq 0$ such that $(Bx_o, Ay_o) \in T_v$, then γ^+ and γ^- exist, and $0 \leq \gamma^+ \leq \gamma^- < \infty$.

Soleimani-Damaneh [8] modified the definitions of Banker's limits by integrating Lemma 1 and foundational notions of returns to scale. The revised definition is as follows:

Definition 2. IRS prevail at (x_o, y_o) if $\gamma^+ > 1$; DRS prevail at (x_o, y_o) if $\gamma^- < 1$; otherwise, CRS prevail at (x_o, y_o).

2.1 Two- stage production process

We examine a production process with a two-stage structure as shown in Fig. 1.

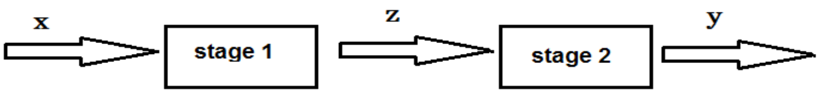

Fig. 1 Two-stage production process

Let us consider a scenario where we have an initial input vector, denoted as x, for the first stage. This vector consists of positive real numbers and has m dimensions ($x \in \mathbb{R}_+^m$). Similarly, we also have a final output vector, represented by y, for the second stage. This vector also consists of positive real numbers and belongs to an S-dimensional real space ($y \in \mathbb{R}_+^S$). Additionally, we possess an intermediate product vector, denoted as z, which serves as both the output of the first stage and the input of the second stage. This vector consists of positive real numbers and resides in a d-dimensional real space ($z \in \mathbb{R}_+^d$).

Suppose we are observing a set of n Decision Making Units (DMUs). For a specific target DMU (DMU_o), we define its input vector, intermediate vector, and final output vector as $x_o = (x_{1o}, x_{2o}, \dots, x_{mo}) > 0$, $z_o = (z_{1o}, z_{2o}, \dots, z_{do}) > 0$, and $y_o = (y_{1o}, y_{2o}, \dots, y_{so}) > 0$, respectively.

The production possibility set of a two-stage production process, where both stages operate independently and do not cooperate, can be expressed as follows:

$$T_\Delta^N = \{(x, z, y) | \quad \sum_{j=1}^n \lambda_j^1 x_j \leq x, \sum_{j=1}^n \lambda_j^1 z_j \geq z, \sum_{j=1}^n \lambda_j^2 z_j \leq z, \sum_{j=1}^n \lambda_j^2 y_j \geq y, \\ \sum_{j=1}^n \lambda_j^1 = \sum_{j=1}^n \lambda_j^2 = \delta, \lambda_j^1 \geq 0, \lambda_j^2 \geq 0, \delta \geq 0, \delta \in \Delta, j = 1, \dots, n\} \tag{3}$$

The value of Δ is determined by the assumption made about the Returns to Scale (RTS) property of the reference technology, and is as follows:

$$\Delta^{VRS} \equiv \{\delta | \delta = 1\}, \quad \Delta^{CRS} \equiv \{\delta | \delta \geq 0\}$$

$$\Delta^{NIRS} \equiv \{\delta | 0 \leq \delta \leq 1\}, \quad \Delta^{NDRS} \equiv \{\delta | \delta \geq 1\}$$

This is a crucial feature of network models, where the production technology's Returns to Scale (RTS) is determined by the parameter δ. In this context, VRS, CRS, NIRS, and

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NDRS represent variable RTS, constant RTS, non-increasing RTS, and non-decreasing RTS, respectively.

To evaluate the efficiency of a particular unit denoted as $DMU_o = (x_o, z_o, y_o)$, where $o \in J$, the input-oriented network radial efficiency measures can be calculated by solving the following linear programming problem:

$$\theta_o^{\Delta} = \min \quad \theta \tag{4}$$

$$s.t \quad \sum_{j=1}^{n} \lambda_j^1 x_j \le \theta x_o \tag{5}$$

$$\sum_{j=1}^{n} \lambda_j^1 z_j \ge z_o \tag{6}$$

$$\sum_{j=1}^{n} \lambda_j^2 z_j \le z_o \tag{7}$$

$$\sum_{j=1}^{n} \lambda_j^2 y_j \ge y_o \tag{8}$$

$$\sum_{j=1}^{n} \lambda_j^1 = \sum_{j=1}^{n} \lambda_j^2 = \delta, \delta \in \Delta \tag{9}$$

$$\lambda_j^1 \ge 0, \lambda_j^2 \ge 0, j = 1, \dots, n \tag{10}$$

where $\Delta \in \{Network_V, Network_C, Network_{NI}, Network_{ND}\}$. We utilize the notation (θ_o^{Δ}) for referring to model (4) under variable RTS assumption.

The output-oriented Network models for assessing DMU_o, which will be used in the sections, are as follows:

$$\varphi_o^{\Delta} = \max \quad \varphi \tag{11}$$

$$s.t \quad \sum_{j=1}^{n} \lambda_j^1 x_j \le x_o \tag{12}$$

$$\sum_{j=1}^{n} \lambda_j^1 z_j \ge z_o \tag{13}$$

$$\sum_{j=1}^{n} \lambda_j^2 z_j \le z_o \tag{14}$$

$$\sum_{j=1}^{n} \lambda_j^2 y_j \ge \varphi y_o \tag{15}$$

$$\sum_{j=1}^{n} \lambda_j^1 = \sum_{j=1}^{n} \lambda_j^2 = \delta, \delta \in \Delta \tag{16}$$

$$\lambda_j^1 \ge 0, \lambda_j^2 \ge 0, j = 1, \dots, n \tag{17}$$

where $\Delta \in \{Network_V, Network_C, Network_{NI}, Network_{ND}\}$. We utilize the notation φ_o^v for referring to model (5) under variable RTS.

We assume that the boundary of T_V^N is denoted by ∂T_V^N. The set of weak network-efficient points in input-orientation (or in output-orientation) is denoted by W_V^I (W_V^O). It should be noted that by the term "boundary," we mean $W_V^I \cup W_V^O$.

Recall that the production point $(x_o, z_o, y_o) \in T_{Nv}$ is identified as a weak network-efficient point in the input orientation when $\theta_{Vo} = 1$, and it is classified as a weak network-efficient point in the output orientation if $\varphi_{Vo} = 1$.

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Definition 3. If there is no $(\hat{x}, \hat{z}_1, \hat{z}_2, \hat{y}) \in T_v^N$ such that $\hat{x} < x_o$, $\hat{z}_1 > z_o$, $\hat{z}_2 < z_o$, and $\hat{y} > y_o$, where the inequalities are understood component-wise, then $DMU_o = (x_o, z_o, y_o)$ is considered network-efficient.

Remark 1. Despite the fact that the definition and results presented in the current work consider network-efficient points, the inefficient units' RTS is regarded as their projection onto the network frontier.

3 The returns to scale of production with two stages

The main aim of this section is to introduce a new notion, NRTS (Network Returns To Scale).

Let $DMU_o = (x_o, z_o, y_o)$ be the unit under consideration. The function $\psi(\alpha)$ corresponding to DMU_o is a function $\psi_o: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by $\psi_o(\alpha) = \frac{\beta_o(\alpha)}{\alpha}$, where:

$$\beta_o(\alpha) = \max \quad \beta \tag{18}$$

$$\begin{aligned} \text{s.t.} \quad & \textstyle\sum_{j=1}^{n} \lambda_j^1 x_j \leq \alpha x_o \\ & \textstyle\sum_{j=1}^{n} \lambda_j^1 z_j \geq \kappa z_o \\ & \textstyle\sum_{j=1}^{n} \lambda_j^2 z_j \leq \kappa z_o \\ & \textstyle\sum_{j=1}^{n} \lambda_j^2 y_j \geq \beta y_o \\ & \textstyle\sum_{j=1}^{n} \lambda_j^1 = \sum_{j=1}^{n} \lambda_j^2 = 1 \\ & \lambda_j^1 \geq 0, \lambda_j^2 \geq 0, \quad j = 1, \dots, n \end{aligned} \tag{19}$$

Certainly, $\beta_o(\alpha)$ denotes the maximum proportion of the output vector y_o that is feasible within the network's production possibility set for the given vector αx_o.

Definition 4. Let $DMU_o = (x_o, z_o, y_o)$ be a network-efficient unit. We say that:

IRS prevails at DMU_o if the maximum $\psi_o(\alpha), \alpha > 0$ happens at $\alpha > 1$

DRS prevails at DMU_o if the maximum $\psi_o(\alpha), \alpha > 0$ happens at $\alpha < 1$

CRS prevails at DMU_o if the maximum $\psi_o(\alpha), \alpha > 0$ happens at least at $\alpha = 1$.

3.1 Motivations

According to model (2) and the condition $\alpha(1) = 1$, we can express the definitions of γ^+ and γ^- for the entire two-stage production process as follows:

$$\gamma^+ = \lim_{\beta \to 1^+} \frac{\alpha(\beta) - 1}{\beta - 1}, \quad \gamma^- = \lim_{\beta \to 1^-} \frac{\alpha(\beta) - 1}{\beta - 1} \tag{20}$$

$\gamma^+ > 1$ means that the rate of increase in outputs to the increase in inputs is greater than one in a right-hand neighborhood of inputs. Also, $\gamma^- < 1$ means that the rate of decrease in

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outputs to the decrease in inputs is less than one in a left-hand neighborhood of inputs; otherwise, CRS prevails at (x_o, z_o, y_o).

Similarly, according to model ?? and the condition $\kappa(1) = 1$, we can express the definitions of γ'^{+} and γ'^{-} for the stage 1 production process as follows:

$$\gamma'^{+} = \lim_{\beta \to 1^{+}} \frac{\kappa(\beta)-1}{\beta-1}, \quad \gamma'^{-} = \lim_{\beta \to 1^{-}} \frac{\kappa(\beta)-1}{\beta-1} \tag{21}$$

$\gamma'^{+} > 1$ means that the rate of increase in intermediates to the increase in inputs is greater than one in a right-hand neighborhood of inputs. Also, $\gamma'^{-} < 1$ means that the rate of decrease in intermediates to the decrease in inputs is less than one in a left-hand neighborhood of inputs; otherwise, CRS prevails at (x_o, z_o, y_o).

Furthermore, according to model (2) and the condition $\alpha(1) = 1$, we can express the definitions of γ''^{+} and γ''^{-} for the stage 2 production process as follows:

$$\gamma''^{+} = \lim_{\kappa(\beta) \to 1^{+}} \frac{\alpha(\beta)-1}{\kappa(\beta)-1}, \quad \gamma''^{-} = \lim_{\kappa(\beta) \to 1^{-}} \frac{\alpha(\beta)-1}{\kappa(\beta)-1} \tag{22}$$

$\gamma''^{+} > 1$ means that the rate of increase in outputs to the increase in intermediates is greater than one in a right-hand neighborhood of inputs. Also, $\gamma''^{-} < 1$ means that the rate of decrease in outputs to the decrease in intermediates is less than one in a left-hand neighborhood of inputs; otherwise, CRS prevails at (x_o, z_o, y_o).

Assumption.

For simplicity, we will use the term "increase rate of output to input" instead of "the rate of increase in the proportion of outputs to the increase in the proportion of inputs in the entire production process" when IRS prevails at DMU_o in the entire production process. Similarly, we will use the term "decrease rate of output to input" instead of "the rate of decrease in the proportion of outputs to the decrease in the proportion of inputs in the entire production process" when DRS prevails at DMU_o in the entire production process.

Also, we will use the term "increase rate of intermediates to input" instead of "the rate of increase in the proportion of intermediates to the increase in the proportion of inputs in stage 1" when IRS prevails at DMU_o in stage 1. Similarly, we will use the term "decrease rate of intermediates to input" instead of "the rate of decrease in the proportion of intermediates to the decrease in the proportion of inputs in stage 1" when DRS prevails at DMU_o in stage 1.

We will use the term "increase rate of output to intermediate" instead of "the rate of increase in the proportion of outputs to the increase in the proportion of intermediates in stage 2" when IRS prevails at DMU_o in stage 2. Similarly, we will use the term "decrease rate of output to intermediate" instead of "the rate of decrease in the proportion of outputs to the decrease in the proportion of intermediates in stage 2" when DRS prevails at DMU_o in stage 2.

Generally, we use the term "the rate of variation." If IRS prevails in the entire production process at DMU_o, this term refers to the "increase rate of output to input." If IRS prevails in stage 1 at DMU_o, this term refers to the "increase rate of intermediates to input." If IRS prevails in stage 2 at DMU_o, this term refers to the "increase rate of output to intermediate." If DRS prevails in the entire production process at DMU_o, it refers to the "decrease rate of output to input." If DRS prevails in stage 1 at DMU_o, it refers to the "decrease rate of intermediates to input." If DRS prevails in stage 2 at DMU_o, it refers to the "decrease rate of output to intermediate."

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Definition 2 states that if the rate of increase of output to input is greater than one in a right-hand neighborhood of inputs, it is recommended to expand the unit's operations. Likewise, if the rate of increase of intermediates to input is greater than one in a right-hand neighborhood of inputs, it is recommended to expand the unit's operations in stage 1. If the rate of increase of output to intermediate is greater than one in a right-hand neighborhood of intermediates, it is recommended to expand the unit's operations in stage 2. If the rate of decrease of output to input is less than one in a left-hand neighborhood of inputs, it is recommended that the unit's operations be reduced. If the rate of decrease of intermediates to input is less than one in a left-hand neighborhood of inputs, it is recommended that the unit's operations in stage 1 be reduced. If the rate of decrease of output to intermediate is less than one in a left-hand neighborhood of intermediates, it is recommended that the unit's operations in stage 2 be reduced.

The purpose of this study is to determine the amount of variation required to achieve the RTS type for the entire two-stage production process and the RTS relationship between the entire production process and the sub-stages for the evaluated unit. This discussion highlights the importance of calculating the variation rate to help managers make informed decisions. To further clarify this concept, consider the following example.

3.2 Motivating example

We have a set of DMUs, represented as $\{DMU_A, DMU_B, DMU_C\}$. Each DMU uses one input to generate one intermediate output and then uses this intermediate output to produce one final output, as shown in Figure 2 and Figure 3.

Table 1 Production data of example 1

DMU	A	B	C
x	2	4	8
z	2	6	8
y	1	9	10

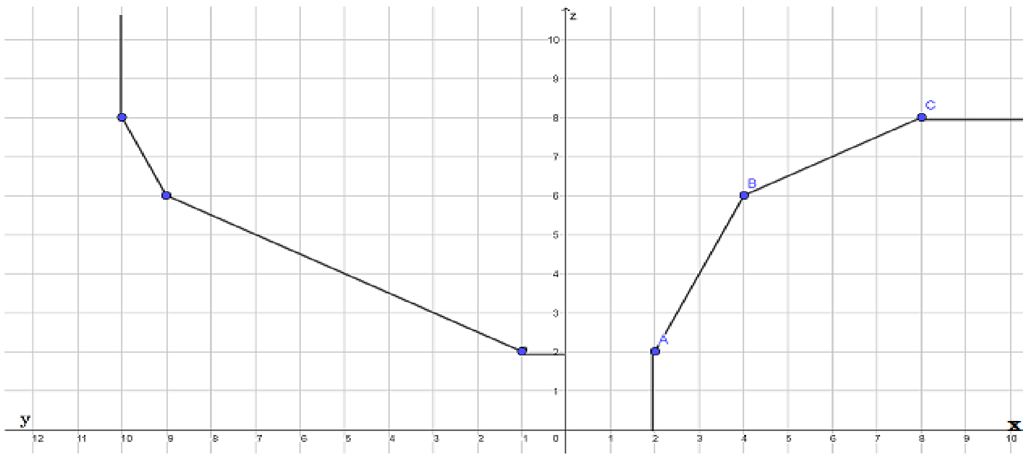

Fig. 2 The PPS of Two-stage production process (numerical example)

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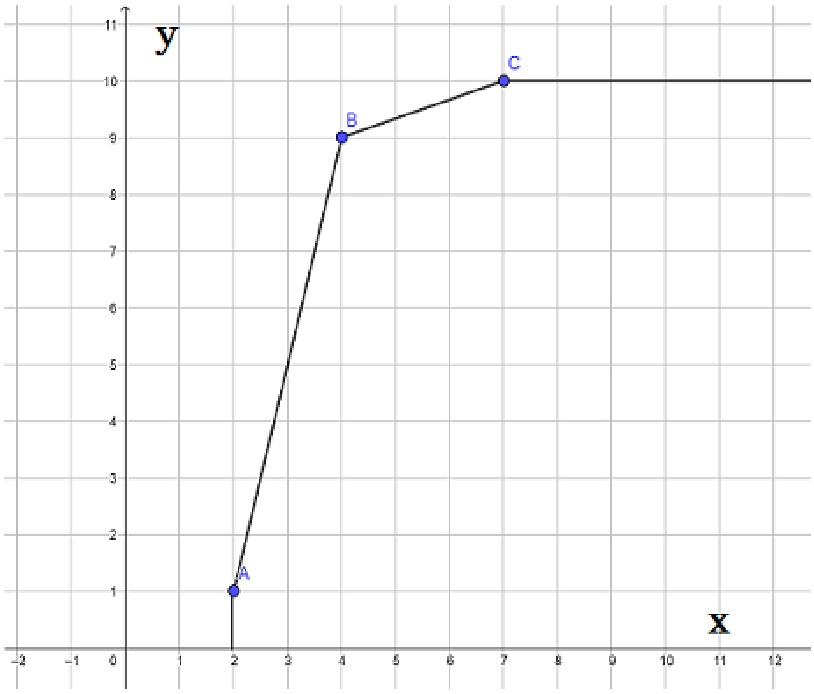

Fig. 3 The PPS of Two-stage production process numerical example in the form of a black box

4 Estimation of the Rate of Variation

To evaluate the returns to scale of each efficient decision-making unit (DMU) in a two-stage network, it is essential to determine the rate of variation for the following three ratios:

1. Final output to input
2. Intermediate output to input
3. Final output to intermediate output

To achieve this, we first define the output-oriented BCC model as follows:

$$\max\{\alpha|\ X\lambda^1 \leq x_o,\ Z\lambda^1 \geq \kappa z_o,\ Z\lambda^2 \leq \kappa z_o,\ Y\lambda^2 \geq \alpha y_o,\ e\lambda^1 = e\lambda^2 = 1,\ \lambda^1 \geq 0,\ \lambda^2 \geq 0\} \tag{23}$$

Assuming that increasing returns to scale (IRS) holds at DMU_o, the right-hand side (RHS) vector of the above model is given by:

$$(x_o, 0,0,0,1,1)^T$$

Now, in the optimal simplex tableau of model (23), we perturb the RHS vector in the direction:

$$(x_o, 0,0,0,0,0)^T$$

which leads to the new RHS vector:

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$$(x_o, 0,0,0,1,1)^T + \delta(x_o, 0,0,0,0,0)^T = ((1+\delta)x_o, 0,0,0,1,1)^T$$

where $\delta \geq 0$ represents the perturbation parameter.

Using parametric analysis, as detailed in Zarepishe and Soleimani (2008), we can determine an interval $[0, \delta^1]$ such that the optimal value of the perturbed problem remains a linear function of δ within this range.

In model (23), the optimal value α indicates the maximum proportional increase in the outputs of DMU_o when all its inputs are scaled by $(1+\delta)$ for $\delta \in [0, \delta^1]$. Thus, the optimal value of the perturbed problem is given by:

$$\alpha(1+\delta), \quad \forall \delta \in [0, \delta^1]$$

Remark 2. Through parametric analysis, both the optimal objective function value and the proportional variation in intermediate outputs as functions of δ are piecewise-linear. Similarly, the optimal objective function value with respect to $\kappa(\delta)$ is also piecewise-linear. However, in this case, we are specifically concerned with its value in the initial interval.

Lemma 2: Let m and m' denote the slopes of $\alpha(1+\delta)$ and $\kappa(1+\delta)$ over the interval $[0, \delta^1]$, respectively. Additionally, let m'' represent the slope of $\alpha(1+\delta)$ over the interval $[0, \kappa(1+\delta)]$. Then, the following rates of variation hold:

- Rate of variation of final output to input: m
- Rate of variation of intermediate output to input: m'
- Rate of variation of final output to intermediate output: m''

These rates are valid when the inputs belong to the interval $[x_o, (1+\delta^1)x_o]$.

Remark 3. The greater the values of m and m', the more cost-effective the increase in outputs and intermediates.

Lemma 3. If m is the slope of $\alpha(1+\delta)$ for $\delta \in [0, \delta^1]$, then γ^+, as defined by (20), at DMU_o is equal to m. If m' is the slope of $\kappa(1+\delta)$ for $\delta \in [0, \delta^1]$, then γ'^+, as defined by (21), at DMU_o is equal to m'. If m'' is the slope of $\alpha(1+\delta)$ for $\kappa \in [0, \kappa(1+\delta)]$, then γ''^+, as defined by (22), at DMU_o is equal to m''.

Corollary. If increasing returns to scale (IRS) prevail at (x_o, z_o, y_o), then $m > 1$. If IRS prevails at (x_o, z_o, y_o) in stage one, then $m' > 1$. If IRS prevails at (x_o, z_o, y_o) in stage two, then $m'' > 1$.

Lemma 4. If IRS prevails at DMU_o and $\hat{\beta} \in [1, 1+\delta^1]$, then the value of γ^+, as defined by equation (20) for $(\beta x_o, \kappa(\beta) z_o, \alpha(\beta) y_o)$, will be greater than one.

So far, we have determined the output-to-input increase rate, the intermediate-to-input increase rate, and the output-to-intermediate increase rate when inputs increase from x_o to $(1+\delta)x_o$. However, our objective is to determine the overall rate of increase to assess the returns to scale (RTS) of the evaluated unit within the entire network process.

If $\gamma^+ \leq 1$, the corresponding point is an MPSS. According to the theorem, when increasing inputs to $((1+\delta^1)x_o, \kappa(\delta^1) z_o, \alpha(\delta^1) y_o)$, there is no need to solve model (23) again. Instead, a modified version of the final simplex tableau from the previous parametric analysis can be utilized.

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Theorem 1. Given the final optimal tableau for model (23), obtained by increasing x_o to $(1+\delta^1)x_o$, we can achieve optimality for the same model by multiplying the objective row and the α-row (excluding the α-column) in the tableau by $\frac{1}{\alpha(\delta^1)}$, and similarly multiplying the κ-row (excluding the κ-column) by $\frac{1}{\kappa(\delta^1)}$. This new tableau corresponds to the values $((1+\delta^1)x_o, \kappa(\delta^1)z_o, \alpha(\delta^1)y_o)$.

While our focus has been on DMUs exhibiting IRS, the same approach can be applied to DMUs classified under DRS. The algorithm detailing this procedure is provided below:

Algorithm.

In order to simplify the algorithm, we set $\delta_0^+ = \delta_0^- = 0$ and $\alpha(0) = 1$.

Step 1. Solve model (23) for $DMU_o = (x_o, z_o, y_o)$, set $i = 1$, and proceed to Step 2.

Step 2. Perform the parametric analysis when the RHS vector is perturbed in the direction of $(x_o, 0,0,0,0,0)^T$. This analysis stops after obtaining the first interval.

Define δ_1^+ as the length of the obtained interval.

Set m_1^+ as the slope of the optimal objective function within this interval.

Set $m_1'^+$ as the slope of the optimal intermediate function within this interval.

Set $m_1''^+$ as the slope of the optimal objective function within the interval from z_o to $\kappa(\delta_1^+)z_o$.

If $m_1^+ > 1$, then IRS prevails in the entire network process (two-stage) at (x_o, z_o, y_o), and proceed to Step 4. Otherwise, if $m_1^+ < 1$, proceed to Step 5. If neither condition holds, CRS prevails in the entire network process (two-stage) at (x_o, z_o, y_o), and proceed to Step 8.

The relationship of RTS between the whole production process and the sub-stages

Based on the above analysis, in this case, the relationship between RTS and the whole production process and substages can be concluded as follows:

As the initial inputs vary, parametric analysis yields an optimal tableau,

(1) The whole process is IRS and

i) Stage 1 is IRS and stage 2 can be IRS or CRS or DRS;

Then $m_1'^+ > 1$, stage 1 is IRS and $m_1''^+ > 1$ or $m_1''^+ \leq 1$, henece Stage 2 is IRS or CRS or DRS at (x_o, z_o, y_o).

ii) Stage 1 is DRS and stage 2 must be IRS;

Then $m_1'^+ < 1$, the stage 1 is IRS and $m_1''^+ > 1$,then the stage 2 is IRS at (x_o, z_o, y_o).

iii) stage 1 is CRS and stage 2 must be IRS.

Then $m_1'^+ \leq 1$, stage 1 is CRS, and $m_1''^+ > 1$,then the stage 2 is IRS at (x_o, z_o, y_o).

Step3. If $m_i^+ \leq 1$, then the CRS prevails at entire process network-two stage at $\left(\left(\prod_{k=0}^{i-1}(1+\delta_k^+)\right)x_o, \left(\prod_{k=0}^{i-1}\kappa(\delta_k^+)\right)z_o, \left(\prod_{k=0}^{i-1}\alpha(\delta_k^+)\right)y_o\right)$, and go step 8; otherwise, proceed to step 4.

The Relationship of RTS Between the Whole Production Process and the Sub-Stages

(2) The entire process is CRS, and:

i) Stage 1 is IRS and stage 2 must be DRS;

If $m_1'^+ > 1$, then stage 1 is IRS, and $m_1''^+ < 1$, then stage 2 is DRS at (x_o, z_o, y_o).

ii) Stage 1 is DRS and stage 2 must be IRS;

If $m_1'^+ < 1$, then stage 1 is DRS, and $m_1''^+ > 1$, then stage 2 is IRS at (x_o, z_o, y_o).

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iii) Stage 1 is CRS and stage 2 must be CRS;
If $m_1'^{+} \leq 1$, then stage 1 is CRS, and $m_1''^{+} \leq 1$, then stage 2 is CRS at (x_o, z_o, y_o).

Step4. In the optimal tableau obtained by the parametric analysis (corresponding to $\left(\left(\prod_{k=0}^{i} (1+\delta_k^{+})\right)x_o, \left(\prod_{k=0}^{i-1} \kappa(\delta_k^{+})\right)z_o, \left(\prod_{k=0}^{i-1} \alpha(\delta_k^{+})\right)y_o\right)$, multiply the cost row and α-row (except for α-column) by $\frac{1}{\alpha(\delta^1)}$ and similarly multiply the κ-row (excluding the κ-column) by $\frac{1}{\kappa(\delta^1)}$ and perform the parametric analysis when the RHS vector is perturbed in the direction of $\left(\left(\prod_{k=0}^{i} (1+\delta_k^{+})\right)x_o, 0,0,0,0,0\right)^T$. In this analysis, stop after obtaining the first interval. Set

δ_{i+1}^{+} = the length of the obtained interval,
m_{i+1}^{+} = the slope of the optimal objective function in the interval obtained in parametric analysis $i = i+1$,and go to step 3.

The relationship of RTS between the entire production process and the sub-stages

In the optimal tableau obtained by the parametric analysis stage 1 (corresponding to $\left(\left(\prod_{k=0}^{i} (1+\delta_k'^{+})\right)x_o, \left(\prod_{k=0}^{i-1} \kappa(\delta_k'^{+})\right)z_o\right)$, multiply the cost row and κ-row (except for κ-column) by $\frac{1}{\kappa(\delta'^1)}$ and do the parametric analysis when the RHS vector is perturbed in the direction of $\left(\left(\prod_{k=0}^{i} (1+\delta_k'^{+})\right)x_o, 0,0\right)^T$ and similarly in the optimal tableau obtained by the parametric analysis stage 2 (corresponding $\left(\left(\prod_{k=0}^{i} (1+\delta_k''^{+})\right)z_o, \left(\prod_{k=0}^{i-1} \alpha(\delta_k''^{+})\right)y_o\right)$ α-row (excluding the α-column) by $\frac{1}{\alpha(\delta''^1)}$ and perform the parametric analysis when the RHS vector is perturbed in the direction of $\left(\left(\prod_{k=0}^{i} (1+\delta_k''^{+})\right)z_o, 0,0\right)^T$. In this analysis, stop after obtaining the first interval.

Set
$\delta_{i+1}'^{+}$ = the length of the obtained interval,
$m_{i+1}'^{+}$ = the slope of the optimal objective function in the interval obtained in parametric analysis.
$\delta_{i+1}''^{+}$ = the length of the obtained interval,
$m_{i+1}''^{+}$ = the slope of the optimal objective function in the interval obtained in parametric analysis.

Step 5. Perform a parametric analysis on problem (23) when the RHS vector is perturbed in the direction of $(-x_o, 0,0,0,0,0)^T$. Stop the analysis after obtaining the first interval.
Define:
δ_1^{-} as the length of the obtained interval,
$m_1^{-} = -1$ (the slope of the optimal objective function within the obtained interval in the parametric analysis).
If $\delta_1^{-} > 0$ and $m_1^{-} < 1$, then DRS prevails at (x_o, z_o, y_o); proceed to Step 7. Otherwise, CRS prevails at (x_o, z_o, y_o); proceed to Step 8.

The relationship of RTS between the whole production process and the sub-stages

(3) The whole process exhibits DRS:

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- Stage 1 exhibits DRS, and Stage 2 can exhibit IRS, CRS, or DRS:
  - If $\delta_1'^- > 0$ and $m_1'^- < 1$, then DRS prevails at (x_o, z_o, y_o) in Stage 1.
  - If $\delta_1''^- > 0$ and $m_1''^- \geq 1$, then IRS or CRS prevails at (x_o, z_o, y_o) in Stage 2.
  - If $\delta_1'^- > 0$ and $m_1'^- < 1$, then DRS prevails at (x_o, z_o, y_o) in Stage 1.
  - If $\delta_1''^- > 0$ and $m_1''^- < 1$, then DRS prevails at (x_o, z_o, y_o) in Stage 2.
- Stage 1 exhibits CRS, and Stage 2 must exhibit DRS:
  - If $\delta_1'^- > 0$ and $m_1'^- > 1$, then IRS or CRS prevails at (x_o, z_o, y_o) in Stage 1.
  - If $\delta_1''^- > 0$ and $m_1''^- < 1$, then DRS prevails at (x_o, z_o, y_o) in Stage 2.
- Stage 1 exhibits IRS, and Stage 2 must exhibit DRS:
  - If $\delta_1'^- > 0$ and $m_1'^- > 1$, then IRS prevails at (x_o, z_o, y_o) in Stage 1.
  - If $\delta_1''^- > 0$ and $m_1''^- < 1$, then DRS prevails at (x_o, z_o, y_o) in Stage 2.

Step 7. In the optimal tableau obtained from the parametric analysis (corresponding to

$$\left(\left(\prod_{k=0}^{i} (1-\delta_k^-)\right)x_o, \left(\prod_{k=0}^{i-1} \kappa(\delta_k^-)\right)z_o, \left(\prod_{k=0}^{i-1} \alpha(\delta_k^-)\right)y_o\right),$$

multiply the cost row and the α-row (except for the α-column) by $\frac{1}{\alpha(\delta_i^-)}$. Similarly, multiply the κ-row (excluding the κ-column) by $\frac{1}{\kappa(\delta_i^-)}$, and perform the parametric analysis when the RHS vector is perturbed in the direction of $\left(-\left(\prod_{k=1}^{i} (1-\delta_k^-)\right)x_o, 0,0,0,0,0\right)^T$. Stop the analysis after obtaining the first interval.

Define:

δ_{i+1}^- as the length of the obtained interval,

$m_{i+1}^- = -1$ (the slope of the optimal objective function within the obtained interval in the parametric analysis).

Set $i = i+1$, and go to Step 6.

Step 8. Finish.

If IRS dominates at (x_o, z_o, y_o) in the aforementioned algorithm, then m_i^+ represents the rate of increase when the inputs change from $\left(\prod_{k=0}^{i-1} (1+\delta_k^+)\right)x_o$ to $\left(\prod_{k=0}^{i} (1+\delta_k^+)\right)x_o$.

On the other hand, if DRS dominates at (x_o, z_o, y_o), then m_i^- indicates the rate of decrease when the inputs change from $\left(\prod_{k=0}^{i-1} (1+\delta_k^-)\right)x_o$ to $\left(\prod_{k=0}^{i} (1+\delta_k^-)\right)x_o$. Otherwise, CRS prevails at (x_o, z_o, y_o).

Hint. The flowchart depicting the algorithm described above can be found in Appendix A. The next remark has a significant practical aspect that can be valuable in integrating previous assessments and provides a beneficial perspective for combining and previous judgments effectively.

Remark 4. Suppose that IRS prevails at (x_o, z_o, y_o). If we aim to increase inputs up to the point where the rate of increase exceeds k_o (where $k_o > 1$ is a manager-specified constant), then Step 3 of the above algorithm can be modified as follows:

Step 3$'$. If $m_i^+ \leq k_o$, then increasing the inputs of

$$\left(\left(\prod_{k=0}^{i-1} (1+\delta^k)\right)x_o, \left(\prod_{k=0}^{i-1} \kappa(\delta^k)\right)z_o, \left(\prod_{k=0}^{i-1} \alpha(\delta^k)\right)y_o\right)$$

is not beneficial, and go to Step 8; otherwise, go to Step 4.

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Suppose that DRS prevails at (x_o, z_o, y_o). If we aim to decrease inputs up to the point where the rate of decrease is less than k_o (where $k_o > 1$ is a manager-specified constant), then Step 6 of the algorithm can be modified as follows:

Step 6′. If $m_i^- \geq k_o$, then decreasing the inputs of

$$\left(\left(\prod_{k=0}^{i-1}(1-\delta_k^-)\right)x_o, \left(\prod_{k=0}^{i-1}\kappa(\delta_k^-)\right)z_o, \left(\prod_{k=0}^{i-1}\alpha(\delta_k^-)\right)y_o\right)$$

is not beneficial, and go to Step 8; otherwise, if $\delta_i^- = 0$, then CRS prevails at this point, and go to Step 8; otherwise, go to Step 7.

5 Numerical Example

This section examines the data from the example in Section 3. We assess DMU_C as the unit under evaluation. Model 23 corresponding to this DMU is as follows:

$$\begin{aligned}
\max \quad & \alpha \\
\text{s.t.} \quad & 2\lambda_1^1 + 4\lambda_2^1 + 8\lambda_3^1 \leq 8 \\
& 2\lambda_1^1 + 6\lambda_2^1 + 8\lambda_3^1 \geq 8\kappa \\
& 2\lambda_1^2 + 6\lambda_2^2 + 8\lambda_3^2 \leq 8\kappa \\
& 1\lambda_1^2 + 9\lambda_2^2 + 10\lambda_3^2 \geq 10\alpha \\
& \lambda_1^1 + \lambda_2^1 + \lambda_3^1 = 1 \\
& \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \\
& \lambda_j^1 \geq 0, \quad \lambda_j^2 \geq 0
\end{aligned}$$

An optimal tableau for this example is:

	Z	λ_1^1	λ_2^1	λ_3^1	λ_1^2	λ_2^2	λ_3^2	κ	α	S_1	S_2	S_3	S_4	R_1	R_2	R_3	R_4	RHS
Z	1	$\frac{3}{10}$	$\frac{1}{10}$	0	$\frac{3}{5}$	0	0	0	0	0	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{10}$	$M-\frac{1}{20}$	$M-\frac{1}{10}$	$M+\frac{2}{5}$	$M+\frac{3}{5}$	1
S_1	0	-6	-4	0	0	0	0	0	0	1	0	0	0	0	0	-8	0	0
λ_3^1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1
α	0	$\frac{3}{10}$	$\frac{1}{10}$	0	$\frac{3}{5}$	0	0	0	1	0	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{10}$	$-\frac{1}{20}$	$-\frac{1}{10}$	$\frac{2}{5}$	$\frac{3}{5}$	1
λ_3^2	0	3	1	0	-2	0	1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	4	-3	1
κ	0	$\frac{3}{4}$	$\frac{1}{4}$	0	0	0	0	1	0	0	$\frac{1}{8}$	0	0	$-\frac{1}{8}$	0	1	0	1
λ_2^2	0	-3	-1	0	3	1	0	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	-4	4	0

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where S_1, S_2, S_3, and S_4 are the variables of the first, second, third, and fourth constraints, respectively. Also, R_1, R_2, R_3, and R_4 are the artificial variables of the second, fourth, fifth, and sixth constraints, respectively. These artificial variables have been kept in the tableau because we need their corresponding columns to obtain B^{-1}.

Now we perform the parametric analysis, when the RHS vector of the problem is perturbed in the direction of $b' = (8,0,0,0,0,0)^T$, then $\bar{b} = (0,1,1,1,1,0)^T$.

The matrix equation for the inverse is:

$$\bar{b}' = B^{-1}b' = \begin{pmatrix} 1 & 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{20} & \frac{1}{20} & -\frac{1}{10} & \frac{2}{5} & \frac{3}{5} \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 4 & -3 \\ 0 & -\frac{1}{8} & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -4 & 4 \end{pmatrix} \begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\delta_1^+ = \min_{i:\bar{b}_i' < 0} \left\{ \frac{\bar{b}_i}{-\bar{b}_i'} \right\} = \infty,$$

$$\alpha(\delta) = c_B\bar{b} + \delta c_B\bar{b}' = 1 + \delta \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1 \Rightarrow m_1^+ = 0.$$

The superscript``$\mathbf{T}''$ stands for transpose. As regards $m_1^+ < 1$, with respect to the algorithm suggested in the previous section, we utilize the parametric analysis when the RHS vector is perturbed in the direction of $b' = (-8,0,0,0,0,0)^{\mathbf{T}}$, and the $\bar{b} = (0,1,1,1,1,0)^{\mathbf{T}}$.

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$$\bar{b}' = B^{-1}b' = \begin{pmatrix} 1 & 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{20} & \frac{1}{20} & -\frac{1}{10} & \frac{2}{5} & \frac{3}{5} \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 4 & -3 \\ 0 & -\frac{1}{8} & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -4 & 4 \end{pmatrix} \begin{pmatrix} -8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since $\bar{b}_1 = 0$ and $\bar{b}_1' < 0$, then performing the algorithm of the parametric analysis in the linear programming, S_1 leaves the basis and λ_2^1 enters the basis by a dual-simplex iteration. Thus, the tableau converts to:

	Z	λ_1^1	λ_2^1	λ_3^1	λ_1^2	λ_2^2	λ_3^2	κ	α	S_1	S_2	S_3	S_4	R_1	R_2	R_3	R_4	RHS
Z	1	$\frac{3}{20}$	0	0	$\frac{3}{5}$	0	0	0	0	$\frac{1}{40}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{10}$	$M-\frac{1}{20}$	$M-\frac{1}{10}$	$M+\frac{1}{5}$	$M+\frac{3}{5}$	$\frac{9}{10}$
λ_2^1	0	$\frac{3}{2}$	1	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	2	0	1
λ_3^1	0	$-\frac{1}{2}$	0	1	0	0	0	0	0	$\frac{1}{40}$	0	0	0	0	0	-1	0	0
α	0	$\frac{3}{20}$	0	0	$\frac{3}{5}$	0	0	0	1	$\frac{1}{40}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{10}$	$-\frac{1}{20}$	$-\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{9}{10}$
λ_3^2	0	$\frac{3}{2}$	0	0	-2	0	1	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	2	-3	0
κ	0	$\frac{3}{8}$	0	0	0	0	0	1	0	$\frac{1}{16}$	$\frac{1}{8}$	0	0	$-\frac{1}{8}$	0	$\frac{1}{2}$	0	$\frac{3}{4}$
λ_2^2	0	$\frac{-3}{2}$	0	0	3	1	0	0	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	-2	4	1

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$$\bar{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \bar{b}' = \begin{pmatrix} \frac{-1}{4} & 0 & 0 & 0 & 2 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & -1 & 0 \\ \frac{1}{40} & \frac{-1}{20} & \frac{1}{20} & \frac{-1}{10} & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{4} & \frac{-1}{2} & \frac{1}{2} & 0 & 2 & -3 \\ \frac{1}{16} & \frac{-1}{8} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{-1}{4} & \frac{1}{2} & \frac{-1}{2} & 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} -8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ \frac{-1}{5} \\ -2 \\ \frac{-1}{2} \\ 2 \end{pmatrix}$$

$$\delta_1^- = \min_{i:\bar{b}_i'<0} \left\{ \frac{\bar{b}_i}{-\bar{b}_i'} \right\} = \frac{1}{2}$$

$$Z = \alpha(\delta) = 1 + \delta(0,0,1,0,0,0) \begin{pmatrix} 2 \\ -2 \\ \frac{-1}{5} \\ -2 \\ \frac{-1}{2} \\ 2 \end{pmatrix} = 1 - \frac{1}{5}\delta \Rightarrow m_1^- = \frac{1}{5}$$

Since $m_1^- = \frac{1}{5} < 1$ and $\delta_1^- > 0$, consequently DRS prevail at C in the entire production process.

The optimal tableau for $\delta = \delta_1^-$ is as follows:

	Z	λ_1^1	λ_2^1	λ_3^1	λ_1^2	λ_2^2	λ_3^2	κ	α	S_1	S_2	S_3	S_4	R_1	R_2	R_3	R_4	RHS
Z	1	$\frac{3}{20}$	0	0	$\frac{3}{5}$	0	0	0	0	$\frac{1}{40}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{10}$	$M - \frac{1}{20}$	$M - \frac{1}{10}$	$M + \frac{1}{5}$	$M + \frac{3}{5}$	$\frac{9}{10}$
λ_2^1	0	$\frac{3}{2}$	1	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	2	0	1
λ_3^1	0	$-\frac{1}{2}$	0	1	0	0	0	0	0	$\frac{1}{40}$	0	0	0	0	0	-1	0	0
α	0	$\frac{3}{20}$	0	0	$\frac{3}{5}$	0	0	0	1	$\frac{1}{40}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{10}$	$-\frac{1}{20}$	$-\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{9}{10}$
λ_3^2	0	$\frac{3}{2}$	0	0	-2	0	1	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	2	-3	0

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κ	0	$\frac{3}{8}$	0	0	0	0	0	1	0	$\frac{1}{16}$	$\frac{1}{8}$	0	0	$-\frac{1}{8}$	0	$\frac{1}{2}$	0	$\frac{3}{4}$
λ_2^2	0	$\frac{-3}{2}$	0	0	3	1	0	0	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	-2	4	1

The relationship of RTS between the whole production process and Stage 1.

Using parametric analysis, the rate of variation intermediate to variation input with respect to $\delta_1^- \in \left[\frac{1}{2}, 1\right]$ is a piecewise-linear function. Regarding $\kappa(\delta_1^-) \in \left[\frac{3}{4}, 1\right]$, the rate of variation intermediate to input is given as follows:

$$\gamma'^- = \lim_{\delta \to 1^-} \frac{\kappa(\delta)-1}{\delta-1} = \frac{\frac{3}{4}-1}{\frac{1}{2}-1} = \frac{\frac{-1}{4}}{\frac{-1}{2}} = \frac{1}{2} \tag{24}$$

This shows that the rate of variation intermediate to input is equal to $\frac{1}{2} < 1$. Therefore, the DRS prevails in DMU_D in Stage 1 within the interval $\delta_1^- \in \left[\frac{1}{2}, 1\right]$.

Also, the rate of variation in output to variation intermediate with respect to $\kappa(\delta_1^-) \in \left[\frac{3}{4}, 1\right]$ in Stage 2 is a piecewise-linear function. Regarding $\alpha(\delta_1^-) \in \left[\frac{1}{9}, 1\right]$, the rate of variation intermediate to input is given as follows:

$$\gamma''^- = \lim_{\kappa(\delta) \to 1^-} \frac{\alpha(\delta)-1}{\kappa(\delta)-1} = \frac{\frac{9}{10}-1}{\frac{3}{4}-1} = \frac{\frac{-1}{10}}{\frac{-1}{4}} = \frac{2}{5} \tag{25}$$

This shows that the rate of variation in output to intermediate is equal to $\frac{1}{2} < 1$. Therefore, the DRS prevails in DMU_C in Stage 2 within the interval $\kappa(\delta_1^-) \in \left[\frac{3}{4}, 1\right]$.

With respect to the algorithm, the cost row and the α-row (excluding the α-column) are multiplied by $\frac{1}{\alpha(\delta_1^-)} = \frac{10}{9}$, and the κ-row (excluding the κ-column) is multiplied by $\frac{1}{\kappa(\delta_1^-)} = \frac{4}{3}$. The following tableau is obtained:

	Z	λ_1^1	λ_2^1	λ_3^1	λ_1^2	λ_2^2	λ_3^2	κ	α	S_1	S_2	S_3	S_4	R_1	R_2	R_3	R_4	RHS
Z	1	$\frac{1}{6}$	0	0	$\frac{2}{3}$	0	0	0	0	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{10}{9}M - \frac{1}{18}$	$\frac{10}{9}M - \frac{1}{9}$	$\frac{10}{9}M + \frac{2}{9}$	$\frac{10}{9}M + \frac{2}{3}$	1
λ_2^1	0	$\frac{3}{2}$	1	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	2	0	1
λ_3^1	0	$-\frac{1}{2}$	0	1	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	-1	0	0
α	0	$\frac{1}{6}$	0	0	$\frac{2}{3}$	0	0	0	1	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{9}$	$-\frac{1}{18}$	$-\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{3}$	1
λ_3^2	0	$\frac{3}{2}$	0	0	-2	0	1	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	2	-3	0
κ	0	$\frac{1}{2}$	0	0	0	0	0	1	0	$\frac{1}{12}$	$\frac{1}{6}$	0	0	$-\frac{1}{6}$	0	$\frac{2}{3}$	0	1
λ_2^2	0	$-\frac{3}{2}$	0	0	3	1	0	0	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	-2	4	1

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Now we utilize parametric analysis in the direction of

$$b' = \begin{pmatrix} -(1-\delta_1^-)8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{, then}$$

$$\overline{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \overline{b}' = \begin{pmatrix} \frac{-1}{4} & 0 & 0 & 0 & 2 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & -1 & 0 \\ \frac{1}{36} & \frac{-1}{18} & \frac{1}{18} & \frac{-1}{9} & \frac{2}{9} & \frac{2}{3} \\ \frac{1}{4} & \frac{-1}{2} & \frac{1}{2} & 0 & 2 & -3 \\ \frac{1}{12} & \frac{-1}{6} & 0 & 0 & \frac{2}{3} & 0 \\ \frac{-1}{4} & \frac{1}{2} & \frac{-1}{2} & 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ \frac{-1}{9} \\ -1 \\ \frac{-1}{3} \\ 1 \end{pmatrix}$$

Since $\overline{b}_2$, $\overline{b}_4$ are equal to zero and $\overline{b}'_2$, $\overline{b}'_4 < 0$, then λ_3^2 leaves the basis and λ_1^2 enters the basis by dual-simplex iteration. Therefore, the tableau converts to:

	Z	λ_1^1	λ_2^1	λ_3^1	λ_1^2	λ_2^2	λ_3^2	κ	α	S_1	S_2	S_3	S_4	R_1	R_2	R_3	R_4	RHS
Z	1	$\frac{2}{3}$	0	0	0	0	$\frac{1}{3}$	0	0	$\frac{1}{9}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{9}$	$\frac{10}{9}M-\frac{2}{9}$	$\frac{10}{9}M-\frac{1}{9}$	$\frac{10}{9}M+\frac{8}{9}$	$\frac{10}{9}M-\frac{1}{3}$	1
λ_2^1	0	$\frac{3}{2}$	1	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	2	0	1
λ_3^1	0	$-\frac{1}{2}$	0	1	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	-1	0	0
α	0	$\frac{2}{3}$	0	0	0	0	$\frac{1}{3}$	0	1	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$-\frac{2}{9}$	$-\frac{1}{9}$	$\frac{8}{9}$	$-\frac{1}{3}$	1
λ_1^2	0	$-\frac{3}{4}$	0	0	1	0	$\frac{-1}{2}$	0	0	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	-1	$\frac{3}{2}$	0
κ	0	$\frac{1}{2}$	0	0	0	0	0	1	0	$\frac{1}{12}$	$\frac{1}{6}$	0	0	$-\frac{1}{6}$	0	$\frac{2}{3}$	0	1

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λ_2^2	0	$\frac{3}{4}$	0	0	0	1	$\frac{3}{2}$	0	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	1	$-\frac{1}{2}$	1

$$\bar{b}=\begin{pmatrix}1\\0\\1\\0\\1\\1\end{pmatrix},\qquad \bar{b}'=\begin{pmatrix}\frac{-1}{4} & 0 & 0 & 0 & 2 & 0\\ \frac{1}{4} & 0 & 0 & 0 & -1 & 0\\ \frac{1}{9} & \frac{-2}{9} & \frac{2}{9} & \frac{-1}{9} & \frac{8}{9} & \frac{-1}{3}\\ \frac{-1}{8} & \frac{1}{4} & \frac{-1}{4} & 0 & -1 & \frac{3}{2}\\ \frac{1}{12} & \frac{-1}{6} & 0 & 0 & \frac{2}{3} & 0\\ \frac{1}{8} & \frac{-1}{4} & \frac{1}{4} & 0 & 1 & \frac{-1}{2}\end{pmatrix}\begin{pmatrix}-4\\0\\0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1\\-1\\\frac{-4}{9}\\\frac{1}{2}\\\frac{-1}{3}\\\frac{-1}{2}\end{pmatrix}$$

Since $\bar{b}_2=0$ and $\bar{b}_2{}'<0$, λ_3^1 is dragged out of the basis and λ_1^1 enters the basis through a dual-simplex iteration. Therefore, the tableau is updated as follows:

	Z	λ_1^1	λ_2^1	λ_3^1	λ_1^2	λ_2^2	λ_3^2	κ	α	S_1	S_2	S_3	S_4	R_1	R_2	R_3	R_4	RHS
Z	1	0	0	$\frac{4}{3}$	0	0	$\frac{1}{3}$	0	0	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{10}{9}M-\frac{2}{9}$	$\frac{10}{9}M-\frac{1}{9}$	$\frac{10}{9}M-\frac{4}{9}$	$\frac{10}{9}M-\frac{1}{3}$	$\frac{1}{9}$
λ_1^2	0	0	1	3	0	0	0	0	0	$\frac{1}{2}$	0	0	0	0	0	2	0	1
λ_1^1	0	1	0	-2	0	0	0	0	0	$-\frac{1}{2}$	0	0	0	0	0	-1	0	0
α	0	0	0	$\frac{4}{3}$	0	0	$\frac{1}{3}$	0	1	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$-\frac{2}{9}$	$-\frac{1}{9}$	$-\frac{4}{9}$	$-\frac{1}{3}$	$\frac{1}{9}$
λ_1^2	0	0	0	$-\frac{3}{2}$	1	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{3}{2}$	1
κ	0	0	0	1	0	0	0	1	0	$\frac{1}{3}$	$\frac{1}{6}$	0	0	$-\frac{1}{6}$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$
λ_2^2	0	0	0	$\frac{3}{2}$	0	1	$\frac{3}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0

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$$\bar{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \bar{b}' = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & -1 & 0 \\ \frac{-1}{2} & 0 & 0 & 0 & 2 & 0 \\ \frac{4}{9} & \frac{-2}{9} & \frac{2}{9} & \frac{-1}{9} & \frac{-4}{9} & \frac{-1}{3} \\ \frac{-1}{2} & \frac{1}{4} & \frac{-1}{4} & 0 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{3} & \frac{-1}{6} & 0 & 0 & \frac{-1}{3} & 0 \\ \frac{1}{2} & \frac{-1}{4} & \frac{1}{4} & 0 & \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ \frac{-16}{9} \\ 2 \\ \frac{-4}{3} \\ -2 \end{pmatrix},$$

$\delta_2^- = \frac{1}{2}$,

$$Z = \alpha(\delta) = 1 + \delta(0,0,1,0,0,0) \begin{pmatrix} -2 \\ 2 \\ \frac{-16}{9} \\ 2 \\ \frac{-4}{3} \\ -2 \end{pmatrix} = 1 - \frac{16}{9}\delta \Rightarrow m_2^- = \frac{16}{9}.$$

Since $m_2^- \geq 1$ and $\delta_2^- > 0$, CRS prevails at DMU_B.

The relationship of RTS between the whole production process and stage 1 is as follows. Using parametric analysis, the rate of variation intermediate to the variation in input with respect to $\delta_2^- \in \left[\frac{1}{2}, 1\right]$ is a piecewise-linear function. Regarding $\kappa(\delta_2^-) \in \left[\frac{1}{3}, 1\right]$, the rate of variation intermediate to input is as follows:

$$\gamma'^- = \lim_{\delta \to 1^-} \frac{\kappa(\delta) - 1}{\delta - 1} = \frac{\frac{1}{3} - 1}{\frac{1}{2} - 1} = \frac{\frac{-2}{3}}{\frac{-1}{2}} = \frac{4}{3}$$

This shows that the rate of variation intermediate to input is equal to $\frac{4}{3} > 1$. Therefore, CRS prevails at DMU_D in stage 1 in the interval $\delta_2^- \in \left[\frac{1}{2}, 1\right]$.

Also, the rate of variation in output with respect to the variation intermediate to $\kappa(\delta_2^-) \in \left[\frac{1}{3}, 1\right]$ in stage 2 is a piecewise-linear function. Regarding $\alpha(\delta_2^-) \in \left[\frac{1}{9}, 1\right]$, the rate of variation intermediate to input is as follows:

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$$\gamma''^{-} = \lim_{\kappa(\delta)\to 1^{-}} \frac{\alpha(\delta)-1}{\kappa(\delta)-1} = \frac{\frac{1}{9}-1}{\frac{1}{3}-1} = \frac{\frac{-8}{9}}{\frac{-2}{3}} = \frac{4}{3}$$

This shows that the rate of variation output to intermediate is equal to $\frac{4}{3} > 1$. Therefore, CRS prevails at DMU_C in stage 2 in the interval $\kappa(\delta_1^-) \in \left[\frac{1}{3}, 1\right]$.

The tableau is updated as follows after multiplying the cost row and α-row (excluding the α-column) by $\frac{1}{\alpha(\delta_2^-)} = \frac{9}{1}$, and the κ-row (excluding the κ-column) is multiplied by $\frac{1}{\kappa(\delta_2^-)} = \frac{3}{1}$:

	Z	λ_1^1	λ_2^1	λ_3^1	λ_1^2	λ_2^2	λ_3^2	κ	α	S_1	S_2	S_3	S_4	R_1	R_2	R_3	R_4	RHS
Z	1	0	0	12	0	0	3	0	0	4	2	2	1	$10M-2$	$10M-1$	$10M-4$	$10M-3$	1
λ_1^2	0	0	1	3	0	0	0	0	0	$\frac{1}{2}$	0	0	0	0	0	2	0	1
λ_1^1	0	1	0	-2	0	0	0	0	0	$-\frac{1}{2}$	0	0	0	0	0	-1	0	0
α	0	0	0	12	0	0	3	0	1	4	2	2	1	-2	-1	-4	-3	1
λ_1^2	0	0	0	$-\frac{3}{2}$	1	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{3}{2}$	1
κ	0	0	0	3	0	0	0	3	0	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	-1	0	1
λ_2^2	0	0	0	$\frac{3}{2}$	0	1	$\frac{3}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0

Now we utilize parametric analysis in the direction of

$$b' = \begin{pmatrix} -(1-\delta_2^-)(1-\delta_1^-)8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{, then}$$

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$$\overline{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \overline{b}' = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 2 & 0 \\ \frac{-1}{2} & 0 & 0 & 0 & -1 & 0 \\ 4 & -2 & 2 & -1 & -4 & -3 \\ \frac{-1}{2} & \frac{1}{4} & \frac{-1}{4} & 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{-1}{2} & 0 & 0 & -1 & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{4} & 0 & \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -8 \\ 1 \\ -2 \\ -1 \end{pmatrix}$$

Since $\overline{b}_6 = 0$ and $\overline{b}'_6 < 0$, then λ_2^2 leaves the basis and there is no variable that enters the basis by dual-simplex iteration. Therefore, the algorithm terminates and the results can be explained as follows:

- If inputs decrease from 8 to $(1 - \delta_1^-)(8) = 4$, then the decrease rate output to input is equal to $m_1^- = \frac{1}{5}$ and the decrease rate intermediate to input is equal to $m_1'^- = \frac{1}{2}$.
  Also, the decrease rate output to intermediate is equal to $m_1''^- = \frac{2}{5}$.
- If inputs decrease from 4 to $(1 - \delta_1^-)(1 - \delta_2^-)(8) = 2$, then the constant rate output to input is equal to $m_1^- = \frac{16}{9}$ and the constant rate intermediate to input is equal to $m_1'^- = \frac{4}{3}$.
  Also, the constant rate output to intermediate is equal to $m_1''^- = \frac{4}{3}$.
- If inputs do not decrease from 2, then the IRS rate output to input is equal to $m_1^- = \infty$ and the IRS rate intermediate to input is equal to $m_1'^- = \infty$.
  Also, the increase rate output to intermediate is equal to $m_1''^- = \infty$.

6 Conclusion

Introduced in 1978 by Charnes et al., Data Envelopment Analysis (DEA) has become a widely recognized and invaluable tool for performance evaluation. Its applications span various sectors, including healthcare, education, banking, and manufacturing, where it supports the assessment of efficiency and effectiveness within specific industries. Over time, DEA has evolved to address complex challenges such as uncertainty, network structures, and multi-stage processes, solidifying its role as a key methodology for performance optimization across diverse fields. Within DEA research, the concept of Returns to Scale (RTS) is essential, offering managers insights to make informed decisions regarding operational strategies. However, traditional RTS analysis mainly provides localized insights and may not fully capture the intricate interactions between inputs, intermediate outputs, and final outputs. This research extends the traditional RTS concept to a two-stage network structure by analyzing rate variations in initial inputs from three perspectives: stage 1, stage 2, and the overall production process. The objective is to investigate how these variations influence the interdependencies among inputs, intermediate outputs, and final outputs within a two-stage

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framework. The structure of a linear programming problem—including parametric extensions—scales linearly with the number of decision variables and constraints. Therefore, when the number of decision-making units (DMUs) or intermediate measures increases, the size of each envelopment model expands proportionally, but its linear nature and tractability remain intact. In other words, our proposed model exactly follows the structure established in the classical theory of parametric linear programming. Modern LP solvers can efficiently handle such enlarged systems, and the interpretation of efficiency measures and returns to scale remains consistent under larger-scale conditions. By employing parametric analysis, this approach provides deeper insights into the interactions between different stages and can be implemented using existing linear programming models, offering a practical tool for decision-makers. In conclusion, this study advances the understanding of RTS in network DEA and offers a robust tool for optimizing production processes. It highlights the importance of considering the structural complexity of production systems when analyzing RTS, reinforcing its role as a flexible and effective approach for performance improvement. Importantly, this model is scalable and performs efficiently even in large-scale problems.

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