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Slack-Based Measurement with Rough Data

S. Kordrostami*, A. Amirteimoori, M. Pakdaman Ashrafiyeh

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Abstract Rough data envelopment analysis (RDEA) evaluates the performance of the decision making units (DMU_s) under rough uncertainty assumption. In this paper, new discussion regarding RDEA is extended. The RSBM model is proposed by integrating SBM model and rough set theory. The process of reaching solution is presented and this model is applied to efficiency evaluation of the DMU_s with uncertain information.

Keywords Data Envelopment Analysis, Uncertainty, Rough Data Envelopment Analysis, Performance Evaluation

1 Introduction

Data envelopment analysis (DEA) is a mathematical programming method for evaluating the relative efficiency of DMU_s with multiple inputs and outputs. DEA presenting the CCR model was introduced by Charnes et al. [1]. Nowadays use of DEA technique is expanding with extra speed. In practical decision making processes, we sometimes encounter uncertainty and factors such as prices, facilities, locations of customers etc. which are changing constantly. One of these uncertainties is rough uncertainty.

The rough set theory was presented by Pawlak [2] and rough variable was presented by Liu [3]. Anyway, little research has been done on rough DEA. Xu et al. [4] proposed the Rough Data Envelopment Analysis (RDEA) and with comparison of RDEA and classic DEA in solving problems with rough uncertainty perceived that RDEA is a more appropriate and exact model.

The remainder of paper is structured as follows: In section 2 concepts such as rough set, rough space, rough variable and trust measure are presented. Rough SBM modeling and its solution are presented in section 3. With a practical example in section 4, the efficiency of RSBM with real data is shown, and section 5 deals with concluding remarks.

2 Definitions and basic concepts

Based on Liu [3], we shall define rough set and rough variable.

* **Corresponding Author.** (✉)
E-mail: kordrostami@liau.ac.ir (S. Kordrostami)

S. Kordrostami, M. Pakdaman Ashrafiyeh
Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran

A. Amirteimoori
Department of Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran

Definition 1. The collection of all sets having the same lower and upper approximations is called a rough set, denoted by $\left(\underline{X}, \overline{X}\right)$.

Definition 2. [4] Let Λ be a nonempty set, A as a σ algebra of subsets of Λ, and Δ an element in A, and π a trust measure. Then, $(\Lambda, \Delta, A, \pi)$ is called a rough space.

Definition 3. [3] Let $(\Lambda, \Delta, A, \pi)$ be a rough space. Then the upper trust of an event κ is defined by $T\overline{r}\{\kappa\} = \dfrac{\pi\{\kappa\}}{\pi\{\Lambda\}}$

The lower trust of the event κ is defined by $T\underline{r}\{\kappa\} = \dfrac{\pi\{\kappa \cap \Delta\}}{\pi\{\Delta\}}$

and the trust of the event κ is defined by $Tr\{\kappa\} = \frac{1}{2}\left(T\overline{r}\{\kappa\} + T\underline{r}\{\kappa\}\right)$.

Definition 4. A rough variable ξ is a measurable function from the rough space $(\Lambda, \Delta, A, \pi)$ to the set of real numbers. That is, for every Borel set B of $\Re$, we have $\left\{\lambda \in \Lambda \middle| \xi(\lambda) \in B\right\} \in A$.
The lower and the upper approximations of the rough variable are defined as follows:
$\underline{\xi} = \left\{\xi(\lambda) \middle| \lambda \in \Delta\right\}$, $\overline{\xi} = \left\{\xi(\lambda) \middle| \lambda \in \Lambda\right\}$
where $\underline{\xi}$ is the lower approximation of the rough variable ξ and $\overline{\xi}$ is the upper approximation of the rough variable ξ.

Definition 5. [5] For the rough value $\xi = ([a,b],[c,d])$, the trust measure of the rough event $\xi \le r$ is defined by

$$\mathrm{Tr}\{\xi \le r\} = \begin{cases} 0 & r \le c \\ \dfrac{1}{2}\left(\dfrac{c-r}{c-d}\right) & c \le r \le a \\ \dfrac{1}{2}\left(\dfrac{c-r}{c-d} - \dfrac{a-r}{a-b}\right) & a \le r \le b \\ \dfrac{1}{2}\left(\dfrac{c-r}{c-d} + 1\right) & b \le r \le d \\ 1 & r \ge d \end{cases}$$

Liu [3] has defined "$\alpha-$pessimistic and $\alpha-$optimistic values of the rough variable" as follows:

Definition 6. Let ξ be a rough variable, and $\alpha \in (0,1]$. Then $\xi_{\sup}(\alpha) = \sup\left\{r \middle| \mathrm{Tr}\{\xi \ge r\} \ge \alpha\right\}$ is called the $\alpha-$optimistic value to ξ, and $\xi_{\inf}(\alpha) = \inf\left\{r \middle| \mathrm{Tr}\{\xi \le r\} \ge \alpha\right\}$ is called the $\alpha-$pessimistic value to ξ.

For example, Let $\xi = ([a,b],[c,d])$ be a rough variable with $c \le a < b \le d$. Then the $\alpha-$optimistic value of ξ is

$$\xi_{\sup}(\alpha)=\begin{cases}(1-2\alpha)d+2\alpha c & \text{if } \alpha\le\dfrac{d-b}{2(d-c)}\\ 2(1-\alpha)d+(2\alpha-1)c & \text{if } \alpha\ge\dfrac{2d-a-c}{2(d-c)}\\ \dfrac{d(b-a)+b(d-c)-2\alpha(b-a)(d-c)}{(b-a)+(d-c)} & \text{otherwise}\end{cases}$$

and the $\alpha-$pessimistic value of ξ is

$$\xi_{\inf}(\alpha)=\begin{cases}(1-2\alpha)c+2\alpha d & \text{if } \alpha\le\dfrac{a-c}{2(d-c)}\\ 2(1-\alpha)c+(2\alpha-1)d & \text{if } \alpha\ge\dfrac{b+d-2c}{2(d-c)}\\ \dfrac{c(b-a)+a(d-c)+2\alpha(b-a)(d-c)}{(b-a)+(d-c)} & \text{otherwise}\end{cases}$$

Theorem 1. Let $\xi_{\inf}(\alpha)$ and $\xi_{\sup}(\alpha)$ be the $\alpha-$pessimistic and $\alpha-$optimistic values of the rough variable ξ, respectively. Then we have

(a) $\mathrm{T}r\{\xi\ge\xi_{\sup}(\alpha)\}\ge\alpha$ and $\mathrm{T}r\{\xi\le\xi_{\inf}(\alpha)\}\ge\alpha$;
(b) $\xi_{\inf}(\alpha)$ is an increasing and left-continuous function of α;
(c) $\xi_{\sup}(\alpha)$ is a decreasing and left-continuous function of α;
(d) if $0<\alpha\le1$, then $\xi_{\inf}(\alpha)=\xi_{\sup}(1-\alpha)$, and $\xi_{\sup}(\alpha)=\xi_{\inf}(1-\alpha)$;
(e) if $0<\alpha\le0.5$, then $\xi_{\inf}(\alpha)\le\xi_{\sup}(\alpha)$;
(f) if $0.5<\alpha\le1$, then $\xi_{\inf}(\alpha)\ge\xi_{\sup}(\alpha)$.

Proof see [3].

3 Rough SBM modeling

Let there are n $\mathrm{DMU_s}$ to be evaluated, each DMU with m inputs and s outputs. Let the inputs and outputs vectors of $\mathrm{DMU_j}$ are rough vectors i.e.

$$\ddot{X}_j=(\ddot{x}_{1j},\dots,\ddot{x}_{mj})^{\mathrm{T}}>0,\quad \ddot{Y}_j=(\ddot{y}_{1j},\dots,\ddot{y}_{sj})^{\mathrm{T}}>0,\quad j=1,\dots,n$$

Assume $\mathrm{DMU_o}$ be the DMU to be evaluated, $\ddot{X}_o$ and $\ddot{Y}_o$ are its input and output vectors. We integrate the SBM model with rough vectors as follows:

$$
\begin{aligned}
Min \quad & \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{\ddot{X}_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s}\frac{t_r^+}{\ddot{Y}_{ro}}} \\
s.t. \quad & \sum_{j=1}^{n}\lambda_j \ddot{X}_{ij} + s_i^- = \ddot{X}_{io}, \qquad i = 1,\dots,m, \\
& \sum_{j=1}^{n}\lambda_j \ddot{Y}_{rj} - t_r^+ = \ddot{Y}_{ro}, \qquad r = 1,\dots,s, \\
& \lambda_j \ge 0,\ s_i^- \ge 0,\ t_r^+ \ge 0, \qquad j = 1,\dots,n.
\end{aligned} \tag{1}
$$

There are plenty of techniques to transfer rough variables into determinate parameters. Liu [3] presented $\alpha-$pessimistic and $\alpha-$optimistic values operator of rough to transfer rough programming into determinate one.

According to the theorem1, if $0.5 < \alpha \le 1$, then $\xi_{\inf}(\alpha) \ge \xi_{\sup}(\alpha)$. Thus, $\alpha-$pessimistic and $\alpha-$optimistic values of the rough variable form interval $\left[\xi_{\sup}(\alpha), \xi_{\inf}(\alpha)\right]$.

Based on the discussion, rough variables $\ddot{X}_j = \left(\ddot{x}_{1j},\dots,\ddot{x}_{mj}\right)^{\mathrm{T}} > 0$ and $\ddot{Y}_j = \left(\ddot{y}_{1j},\dots,\ddot{y}_{sj}\right)^{\mathrm{T}} > 0$ are transformed into intervals $\left[X_j^{\sup(\alpha)}, X_j^{\inf(\alpha)}\right]$ and $\left[Y_j^{\sup(\alpha)}, Y_j^{\inf(\alpha)}\right]$, respectively. Therefore, the model (1) is transformed into the interval programming (2):

$$
\begin{aligned}
Min \quad & \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{\left[X_{io}^{\sup(\alpha)}, X_{io}^{\inf(\alpha)}\right]}}{1 + \frac{1}{s}\sum_{r=1}^{s}\frac{t_r^+}{\left[Y_{ro}^{\sup(\alpha)}, Y_{ro}^{\inf(\alpha)}\right]}} \\
s.t. \quad & \sum_{j=1}^{n}\lambda_j \left[X_{ij}^{\sup(\alpha)}, X_{ij}^{\inf(\alpha)}\right] \le \left[X_{io}^{\sup(\alpha)}, X_{io}^{\inf(\alpha)}\right], \\
& \sum_{j=1}^{n}\lambda_j \left[Y_{rj}^{\sup(\alpha)}, Y_{rj}^{\inf(\alpha)}\right] \ge \left[Y_{ro}^{\sup(\alpha)}, Y_{ro}^{\inf(\alpha)}\right], \\
& \lambda_j \ge 0, \quad j = 1,\dots,n.
\end{aligned} \tag{2}
$$

In order to transform the programming based on function (2) into linear exact programming under the known certainty level of a, if the inputs and outputs of DMU_o be the minimum and maximum values respectively, and if the inputs and outputs of the rest of the n-1 DMU_s be the maximum and minimum values respectively, the programming based on model (2) is transformed into linear programming (3) from which the maximum efficiency point is obtained presented as $\rho^{\inf(\alpha)}$.

$$Min \quad \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{X_{io}^{\sup(\alpha)}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{t_r^+}{Y_{ro}^{\inf(\alpha)}}}$$

$$\begin{aligned} s.t. \quad & \sum_{j=1(j \neq jo)}^{n} \lambda_j X_{ij}^{\inf(\alpha)} + \lambda_{jo} X_{io}^{\sup(\alpha)} + s_i^- = X_{io}^{\sup(\alpha)}, \qquad i = 1,\ldots,m, \\ & \sum_{j=1(j \neq jo)}^{n} \lambda_j Y_{rj}^{\sup(\alpha)} + \lambda_{jo} Y_{ro}^{\inf(\alpha)} - t_r^+ = Y_{ro}^{\inf(\alpha)}, \qquad r = 1,\ldots,s, \\ & \lambda_j \geq 0,\ s_i^- \geq 0,\ t_r^+ \geq 0, \quad j = 1,\ldots,n. \end{aligned} \tag{3}$$

Now, if the inputs and outputs of DMU_o be the maximum and minimum values, respectively, and if the inputs and outputs of the rest of the n-1 DMU_s be the minimum and maximum values, respectively, the programming based on model (2) is transformed into linear programming (4) from which the minimum efficiency point is obtained presented as $\rho^{\sup(\alpha)}$

$$Min \quad \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{X_{io}^{\inf(\alpha)}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{t_r^+}{Y_{ro}^{\sup(\alpha)}}}$$

$$\begin{aligned} s.t. \quad & \sum_{j=1(j \neq jo)}^{n} \lambda_j X_{ij}^{\sup(\alpha)} + \lambda_{jo} X_{io}^{\inf(\alpha)} + s_i^- = X_{io}^{\inf(\alpha)}, \qquad i = 1,\ldots,m, \\ & \sum_{j=1(j \neq jo)}^{n} \lambda_j Y_{rj}^{\inf(\alpha)} + \lambda_{jo} Y_{ro}^{\sup(\alpha)} - t_r^+ = Y_{ro}^{\sup(\alpha)}, \qquad r = 1,\ldots,s, \\ & \lambda_j \geq 0,\ s_i^- \geq 0,\ t_r^+ \geq 0, \quad j = 1,\ldots,n. \end{aligned} \tag{4}$$

Models (3) and (4) are having deterministic parameters and $\left(\rho^*\right)^{\inf(\alpha)} \geq \left(\rho^*\right)^{\sup(\alpha)}$.

Definition 7. DMU_o is RSBM efficient if $\left(\rho^*\right)^{\inf(\alpha)} = 1$, otherwise, it is RSBM inefficient.

4 Practical application

In this section, the model proposed is applied to a real data on 8 bank branches. The data are derived from operations during the years of 1387 and 1388. The inputs include the number of staff, amount of demands and outputs are amount of deposits, amount of loans. Except the number of staff, the other data are considered as rough. Based on Xu et al. [4], rough variables are made as follows:

As an example, let n be the amount of demands of the branch in 1387 and let m be the amount of demands of the same branch in 1388. Without loss of generality, suppose $\mathrm{n} \prec \mathrm{m}$, then the interval $[n,m]$ is considered as the upper approximation of a rough variable and let $p=(1/2)(m+n)$, the lower approximation of a rough variable can be considered as an interval nearby p.

In the table (1), X_1 denotes number of staff, X_2 denotes amount of demands, Y_1 denotes amount of deposits and Y_2 denotes amount of loans.

Table 1 Input and output data of DMU_s

DMU	X_1	X_2	Y_1	Y_2
1	12	([5900,6100],[4308,7699])	([60000,61000],[58635,62310])	([91000,93000],[88885,95354])
2	6	([7800,7870],[7648,8071])	([28800,29000],[27756,29891])	([41290,41300],[41247,41344])
3	4	([920,950],[862,999])	([7170,7180],[7109,7241])	([22870,22900],[20558,25195])
4	4	([1900,1910],[1804,2002])	([11800,11900],[11495,12208])	([25600,25700],[23400,27860])
5	3	([1440,1450],[1333,1548])	([6200,6400],[5925,6710])	([17000,18000],[15808,20140])
6	7	([680,690],[589,783])	([8500,8600],[7972,9112])	([42500,42800],[41627,43672])
7	3	([2900,2940],[2826,3045])	([9900,9990],[9803,10095])	([20400,21000],[17957,22854])
8	5	([1200,1280],[226,2315])	([49200,49300],[49049,49476])	([61000,63000],[58288,66057])

By solving the two models (3) and (4), the efficiency interval of DMU_o is gotten, which is denoted as $\left[\rho^{\sup(\alpha)},\rho^{\inf(\alpha)}\right]$. Table 2 shows the evaluation results under trust level $\alpha=0.9$.

Table 2 The results of RSBM model

DMU	Efficiency interval
1	[0.3285,0.5556]
2	[0.2756,0.3461]
3	[0.1927,0.3645]
4	[0.2287,0.3621]
5	[0.1747,0.2871]
6	[0.2089,1.0000]
7	[0.2235,0.3049]
8	[1.0000,1.0000]

Based on Definition 7, DMU_6 and DMU_8 are RSBM efficient and their slacks are zero, too.

5 Conclusions

In this paper, a DEA model with rough parameters has been presented. In the process of solving the RSBM model, the $\alpha-$ pessimistic and $\alpha-$optimistic values of the rough variable to transfer the rough model into deterministic linear programming were employed. The efficiency interval $[\rho^{\sup(\alpha)},\rho^{\inf(\alpha)}]$ was considered efficiency score of each DMU. In a practical example with real data the proposed model has been studied. At the end, it should be noted that research on the combination of other DEA models with rough set theory will provide interesting research areas.

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