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Simulation of Characteristic of Linear Degradation with Failure Modes

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Abstract In this paper, we consider linear degradation in the case of multiple failure modes and obtain the graph of survival function, traumatic failure probability function, natural failure probability function by simulating data. We compare traumatic failure probability function, natural failure probability function and survival probability in different sample sizes. Also sample path is simulated by using noise.

Keywords Degradation Process, Critical Level, Failure Data, Natural Failure, Traumatic Failure, Non-parametric Estimation

1 Introduction

One way of obtaining additional information about reliability of units can be degradation. In 1998, Meeker and Scobar [1] expressed applicative degradation models in engineering affaires in their book. Nikulin and Bagdonavicius presented accelerated degradation models [2]. Degradation models with the explanatory variables (controllable and uncontrollable covariates) are used to estimate reliability [3,4]. For example, tire wear rate and failure times depend on quality of roads, temperature and other factors (as uncontrollable covariates) and a puncture of a tire and the load (as controllable covariates) [5,6,7].

Non-parametric methods for linear degradation–failure time data without renewals are given in Bagdonavicius et al. [7]. After that they could extend their work and found non parametric estimators of intensities and failure probabilities with partial renewals [5]. Kahle obtained statistical analysis of degradation failure time renewal data by using parametric methods [8].

2 The model

Now we recall some notions of this model from Bagdonavicius et al. [7]. Let $Z(t)$ denote the degradation process value at the t moment and a unit is renewed when its degradation attains

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some critical level z_0. For $j \geq 1$, let S_j denote the moment of the j th renewal (we assume $(S_1 = 0)$) and degradation is linear in the interval $(S_j; S_{j+1}]$ with random degradation rate [1]. A_j is the inverse of the degradation rate in this interval. Suppose that the random variables $A_1, A_2, \ldots$ are independent and positive with the cumulative distribution functions $\pi_1, \pi_2, \ldots$.

$T^{(k)}$ $(k = 1, \ldots, s)$ is the moment of k th traumatic failure mode. Let $\widetilde{T} = \min(T^{(1)}, \ldots, T^{(s)})$ denote the moment of a traumatic failure.

Degradation process model is introduced by:

$$Z(t) = (t - S_j) / A_j \quad for \quad S_j < t \leq S_{j+1} \tag{1}$$

where

$$S_{j+1} = \sum_{i=1}^{j} A_i z_0 .$$

The random variables $T^{(1)}, \ldots, T^{(S)}$ are conditionally independent and have the intensities depending only on the wear level. It means that the conditional survival function of $T^{(k)}$ is

$$S^{(k)}(t|Z(s)) = P(T^{(k)} > t \mid Z(s), 0 \leq s \leq t) = \exp\{-\int_0^t \lambda^{(k)}(Z(s))ds\} \tag{2}$$

where

$$\Lambda(z) = \sum_{k=1}^{s} \Lambda^{(k)}(z) \qquad \lambda(z) = \sum_{k=1}^{s} \lambda^{(k)}(z), \qquad \Lambda^{(k)}(z) = \int_0^z \lambda^{(k)}(y)dy,$$

So

$$S(t|Z(s)) = P(\widetilde{T} > t \mid Z(s). 0 \leq s \leq t) = \exp\{-\int_0^t \lambda(Z(s))ds\}$$

Formula (2) also implies that the conditional probability density function of $T^{(k)}$ is

$$f^{(k)}(t|Z(s)) = \lambda^{(k)}(Z(s))(\exp\{-\int_0^t \lambda^{(k)}(Z(s))ds\}) \tag{3}$$

So

$$f(t|Z(s)) = \lambda(Z(s))(\exp\{-\int_0^t \lambda(Z(s))ds\})$$

Then, the survival function of the random variable T equals respectively,

$$S(t) = \int_{a_j}^{\infty} \exp\{-a\Lambda(z)\} d\pi_j(a), \tag{4}$$

Denote by $\widetilde{V}$ the indicator of the failure mode that shows what kind of failure mode has occurred for considered item:

$$\widetilde{V} = \begin{cases} 0 & if\ T = T^{(0)} \\ 1 & if\ T = T^{(1)} \\ \vdots & \\ k & if\ T = T^{(k)} \end{cases}$$

The important reliability characteristics are the probability $p_j^{(k)}(z)$ of the failure of the kth mode that a failure of the mode occurs before the level of degradation attains the level z given that a unit had been renewed $(j-1)$ times.

$$p_j^{(k)}(z) = P(\widetilde{T} \le S_j + zA_j, V = k \mid \widetilde{T} > S_j)$$
$$= \int_0^\infty a \int_0^z \exp\{-a\Lambda(y)\} d\Lambda^{(k)}(y) d\pi_j(a), \tag{5}$$

Adding $(k = 1, \dots, s)$ up, we obtain the probability $p_j(z)$ of a traumatic failure that a failure occurs before the level of degradation attains the level $z.(0 \le z \le z_0)$ given that a unit had been renewed $(j-1)$ times.

$$p^{(tr)}{}_j(z) = P(\widetilde{T} \le S_j + zA_j \mid \widetilde{T} > S_j)$$
$$= 1 - \int_0^\infty \exp\{-a\Lambda(z)\} d\pi_j(a), \tag{6}$$

Formulas (4) and (6) imply that the probability of a natural failure in the interval $[0; t]$ is

$$P^{(0)}(t) = 1 - S(t) - P^{(tr)}(t) = \int_0^{a_j} \exp\{-a\Lambda(z)\} d\pi_j(a), \tag{7}$$

Now suppose that the each item's curve is known up to t moment, set:

$$z = g(t,a) = (t - S_j)/A_j \quad , \quad z_1 = g(t + \Delta, a)$$

Denote in the interval $[t, t+\Delta]$, the probability of a failure of the kth mode and the probability of a traumatic failure in the same interval, respectively, by $Q^{(k)}(\Delta)$, $Q^{(tr)}(\Delta)$. With this notation we get

$$Q^{(k)}(\Delta) = a\int_z^{z_1} e^{-a[\Lambda(y) - \Lambda(z)]} d\Lambda^{(k)}(y)$$
$$Q^{(tr)}(\Delta) = 1 - e^{-a[\Lambda(y) - \Lambda(z)]}$$

3 Non-parametric estimation of the cumulative intensities $\Lambda^{(k)}$

The cumulative intensities $\Lambda^{(k)}$ are very important; therefore, we assign $\Lambda^{(k)}$ to this characteristics.

$(S_1, \dots . S_m, T, Z(T), V)$.

If n units are to be tested, then for the i^th unit we can be defined as the following collection of vectors of a random length:

$(S_{i1}, \dots \dots, S_{imi}, T_i, Z_i, V_i), i = 1, \dots, n.$

The cumulative distribution functions π_j are estimated by

$$\hat{\pi}_j(a) = \frac{\sum_{i=1}^{n} 1_{\{A_{ij} \le a, m_i \ge j\}}}{m(j)}, m(j) = \sum_{i=1}^{n} 1_{\{j \le m_i\}} \tag{8}$$

For $z \in [0; z_0), k = 1, \ldots, s$ Set

$$N(z) = 1_{\{Z(T) \le z. T = \tilde{T}\}}, \qquad N^{(k)}(z) = 1_{\{Z(T) \le z, V = K\}} \tag{9}$$

The process $N^{(k)}(z)$ can be written as the sum

$$\mathrm{N}^{(k)}(z) = \int_0^z Y(y) d\Lambda^{(k)}(y) + M^{(k)}(z), \tag{10}$$

where

$$Y(y) = \frac{A_m 1_{\{Z(T) \ge y\}}}{1 - e^{-A_m(\Lambda(z_0) - \Lambda(y)) - \tilde{\Lambda}^{(0)}(S_{m+1}) + \tilde{\Lambda}^{(0)}(S_m + A_m y)}}$$

and $M^{(k)}(z)$ is a martingale with respect to the filtration $(F_z | 0 \le z \le z_0)$ and F_z denote the σ-algebra generated by the following collections of events:

$$\{A_1 \le a_1, ...A_j \le a_j\} \cap \{m = j\} \tag{11}$$

and

$$\{A_1 \le a_1, ...A_j \le a_j\} \cap \{m = j\} \cap \{Z(T) \le y, V = k\}. \tag{12}$$

where $j \ge 1, k = 1, ..., s, a_1, ..., a_j > 0$ & $y \le z$ [2,9].

Let us consider another decomposition of the processes $N(z)$ and $N^{(k)}(z)$ Set

$$N^*(t) = 1_{\{T \le t. T = \tilde{T}\}}, N_k^*(t) = 1_{\{T \le t, V = k\}}, Y^*(t) = 1_{\{T \ge t\}}.$$

Denote by $F^*{}_t$ the σ _algebra generated by $N_k^*(s), Y^*(s), 0 \le s \le t, k = 1, \ldots, s$. So

$$N_k^*(t) = \int_0^t \lambda^{(k)}(Z(u)) Y^*(u) du + M_k^*(u) \tag{13}$$

where $M_k^*(u)$ is a martingale with respect to the filtration $(F_t^* | t \ge 0)$. Set

$$Z = Z(T), Z_j = \begin{cases} z_0 & if\ j < m \\ Z & if\ j = m \end{cases} \tag{14}$$

Also, the process $N^{(k)}(z)$ can be written as the sum

$$N^{(k)}(z) = \int_0^z Y_k^{**}(y) d\Lambda^{(k)}(y) + M_k^{**}(z) \tag{15}$$

where

$$Y_k^{**}(y)=\sum_{j=1}^{m}A_j 1_{\{zj\geq y\}},\quad M_K^{**}(z)=\int_0^\infty 1_{\{z(u)\leq z\}}\,dM_k^*(u) \tag{16}$$

Now define the processes $N_i^{(k)}(z)$, $Y_i(y)$ and $M_i^{(k)}(z)$ as in first decomposition, with $Z(T), A_m, S_m, V$ replaced by $Z_i(T_i), A_{im_i}, S_{im_i}, V_i$ .Set

$$\overline{N}^{(k)}(z)=\sum_{i=1}^{n}N_i^k(z),\quad \overline{Y}(y)=\sum_{i=1}^{n}Y_i(y),\quad \overline{M}^{(k)}(z)=\sum_{i=1}^{n}M_i^{(k)}(z),$$

Another decomposition of process $\overline{N}^{(k)}(z)$ can be writing as the sum

$$\overline{N}^{(k)}(z)=\int_o^z \widetilde{Y}(y)d\Lambda^{(k)}(y)+\widetilde{M}^{(k)}(z). \tag{17}$$

where

$$\widetilde{Y}(y)=\sum_{i=1}^{n}\sum_{j=1}^{m_i}A_{ij}1\{Z_{ij}\geq y\},\qquad \widetilde{M}^{(k)}(z)=\sum_{i=1}^{n}\int_o^\infty 1\{Z_i(u)\leq z\}dM_{ki}^*(u)$$

Moreover, M_{ki}^* is a martingale with respect to the filtration:

$$F=\{F_{it}^*=\sigma(N_i^*(s),Y_i^*(s),0\leq s\leq t)\big|t\geq 0\},\qquad N_i^*(t)=1\{T_i\leq t,V_i=k\},Y_i^*(t)=1\{T_i\geq t\}.$$

The decomposition (17) implies the estimator

$$\hat{\Lambda}^{(k)}(z)=\int_o^z \widetilde{Y}^{-1}(y)d\overline{N}^{(k)}(y)=\sum_{Z_k\leq z,v\neq 0}\widetilde{Y}^{-1}(Z_k)=\sum_{Z_k\leq z,v\neq 0}\left(\sum_{i=1}^{n}\sum_{j=1}^{m_i}A_{ij}1\{Z_{ij}\geq Z_k\}\right)^{-1}$$

4 The estimation of the probabilities $p_j(z)$ and $p_j^{(k)}(z)$

If $\pi_j=\pi$ then the probability $p_j(z)$ to failure before the degradation attains the level $z.(0\leq z\leq z_0)$ given that a unit had been renewed $(j-1)$ times, is estimated by the statistic:

$$\hat{p}^{(tr)}{}_j(z)=1-\frac{1}{m}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\exp\left\{-A_{ij}\sum_{k;Z_k\leq z,V_k\neq o}1/\widetilde{Y}(Z_k)\right\}$$

Since

$$m=\sum_{i=1}^{n}m_i.\qquad \hat{\pi}(a)=\frac{1}{m}\sum_{i=1}^{n}\sum_{j=1}^{m_i}1_{\{A_{ij}\leq a\}}$$

Otherwise, the estimator is

$$\hat{p}^{(tr)}{}_j(z)=1-\frac{1}{m(j)}\sum_{m_i\geq j}\exp\left\{-A_{ij}\sum_{k;Z_k\leq z,V_k\neq o}1/\widetilde{Y}(Z_k)\right\}$$

The estimator of the probability $p_j^k(z)$

$$p_j^k(z) = \frac{1}{m(j)} \sum_{i:m_i \geq j} \sum_{l;Z_l \leq z, V_l = k} A_{ij} \exp\left\{ -A_{ij} \sum_{s;Z_s \leq Z_l, V_s \neq o} 1/\widetilde{Y}(Z_s) \right\} \frac{1}{Y(Z_l)}$$

The estimators $\hat{p}_j$ and $\hat{p}^{(k)}{}_j$ are functions of the estimators $\hat{\Lambda}^{(k)}$ and $\hat{\pi}_j$.

5 Simulation in non-parametric method

For analyzing the application of the received model, we apply the simulation method. This simulation is done by the software R. In 2000, Professor Bagdonavicius researched degradation value and failure time of 101bus tire. The critical level of degradation in that information was 15mm. In addition, the traumatic failure can be categorized in two groups: the first group is the tire that punctured and the second group is the tire that exploded. In this part, the graph of the path of item, survival function, traumatic failure probability and natural failure probability functions are produced by using simulation.
Suppose that the model of linear degradation process is:

$$Z(t) = \frac{t}{A}$$

The simulation of the path of the item has linear model, and the natural failure value is considered by $Z = \frac{T}{A}$. Suppose that an item failure takes place, when its degradation reaches the critical level z_0 or a traumatic failure occurs. So, traumatic failure occurs for observed degradation paths when the last degradation value is less than $Z_0 = 15$.

Suppose $T^{(0)}$ is the natural failure time. So natural failure time of ith item, meaning $T^{(0)} = z_0 A$ is $T^{(0)} = 15 \times A_i$. Consider the vector $A = (A_1, A_2)$ effected degradation and has bivariate normal distribution that $A \sim \mathrm{N}(\mu, \Sigma)$
where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \text{And} \quad \Sigma = \begin{pmatrix} \sigma_1{}^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2{}^2 \end{pmatrix}$$

In the software R is not possible to produce any random variable with bivariate normal distribution; therefore, to produce variable A we use transform $B = A\Sigma^{-1/2}$.

With this transformation, vector $B = (B_1, B_2)$ has bivariate normal distribution with the mean $\mu' = \mu\Sigma^{-1/2}$ and the variance $\Sigma' = Var(A\Sigma^{-1/2})$.
Assume that $A = (A_1, A_2) = N(1.8\,, 2.3\,, 0.2\,, 0.2\,, 0)$ we will have:

$$\mathrm{B}_2 \sim \mathrm{N}\left(\frac{2.3}{\sqrt{0.2}}, 1\right) \qquad \mathrm{B}_1 \sim \mathrm{N}\left(\frac{1.8}{\sqrt{0.2}}, 1\right)$$

For producing failure time, we use a parametric risk function and with survival conditional function, we produce every failure times. The intensity functions of these failures are $\lambda^{(k)}(z) = (\frac{z}{\theta})^{v}$,so we have $T_i = [-(\frac{A_i}{\theta})(\alpha + v + 1)\ln S_T]^{\frac{1}{\alpha+v+1}}$

where
$S_T \sim U(0,1)$

In this simulation, we consider $(\alpha , v , \theta) = (0 , 4.7 , 130)$ and we produce Z (t) by the calculated; T, A. Also for each i we calculate T_1 and T_2 . Finally each item failure time is defined by $\min(T_0, T_1, T_2)$.

In figure 1: The graph of survival function for 100 simulated items is drawn.

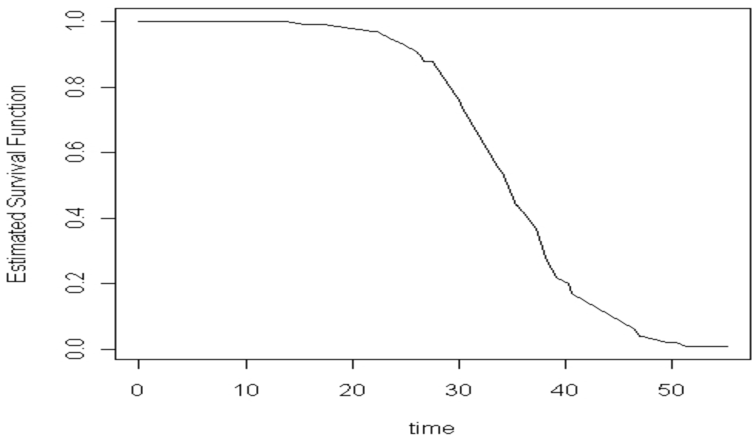

Fig. 1 The graph of survival function

In figure 2 and figure 3: The graph of 100 simulated items for traumatic failure probability and natural failure probability functions are drawn.

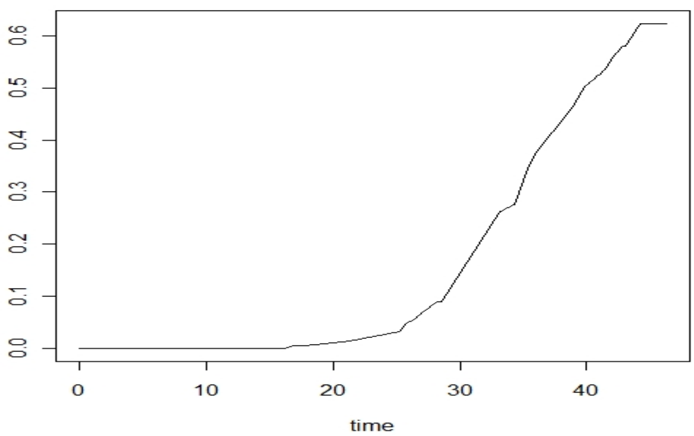

Fig. 2 The graph of traumatic failure probability function

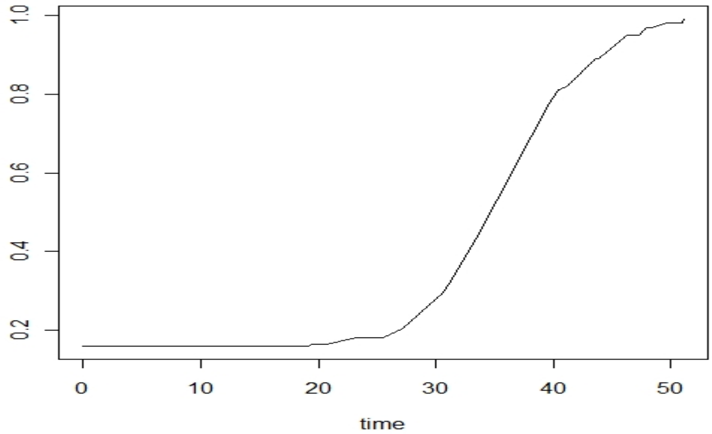

Fig. 3 The graph of natural failure probability function

In figure 4: The graph of traumatic failure probability functions is drawn with different sample sizes. We can observe when sample size is increasing, the probability function graph becomes more natural and their slope comes close to normal (Traumatic failure probability function, with n=10 is shown with dots, with n=40 is shown with broken lines and with n=100 is bold).

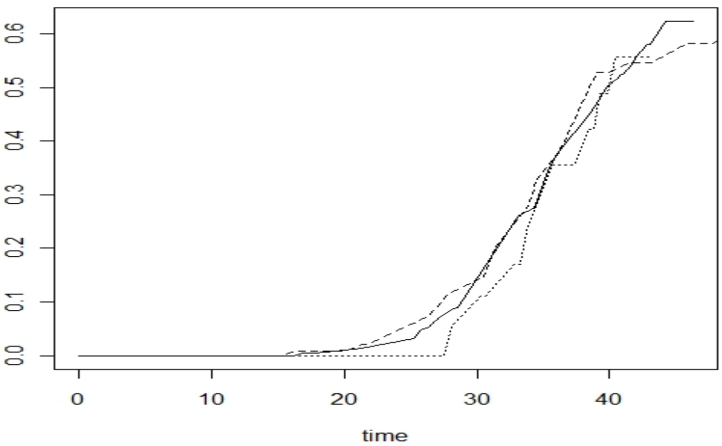

Fig. 4 Comparison of traumatic failure probability function with different sample sizes

In figure 5: The graph of natural failure probability functions is drawn with different sample sizes. We can observe when sample size is increasing the probability function graph becomes more natural and their slope comes close to normal. (Natural failure probability function, with n=10 is shown with dots, with n=40 is shown with broken lines and with n=100 is bold)

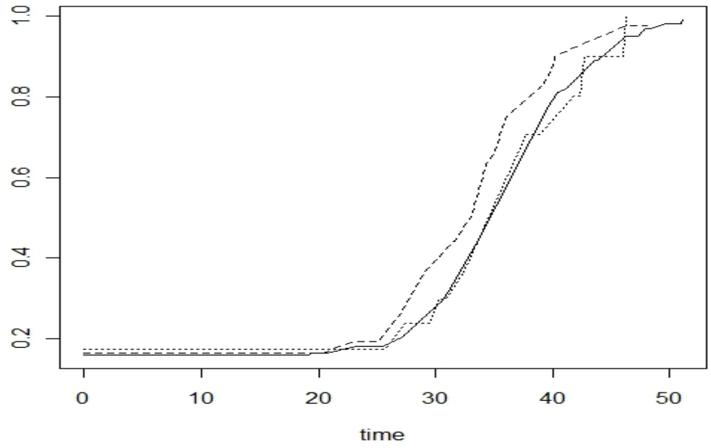

Fig. 5 Comparison of natural failure probability function with different sample sizes

In figure 6: The graph of survival function is drawn with different sample sizes. We can observe when sample size is increasing; the survival function graph becomes more natural. So their curve becomes smoother. According to the survival function feature that have decreasing slope, at first function work with the probability 1 and finally comes to 0.

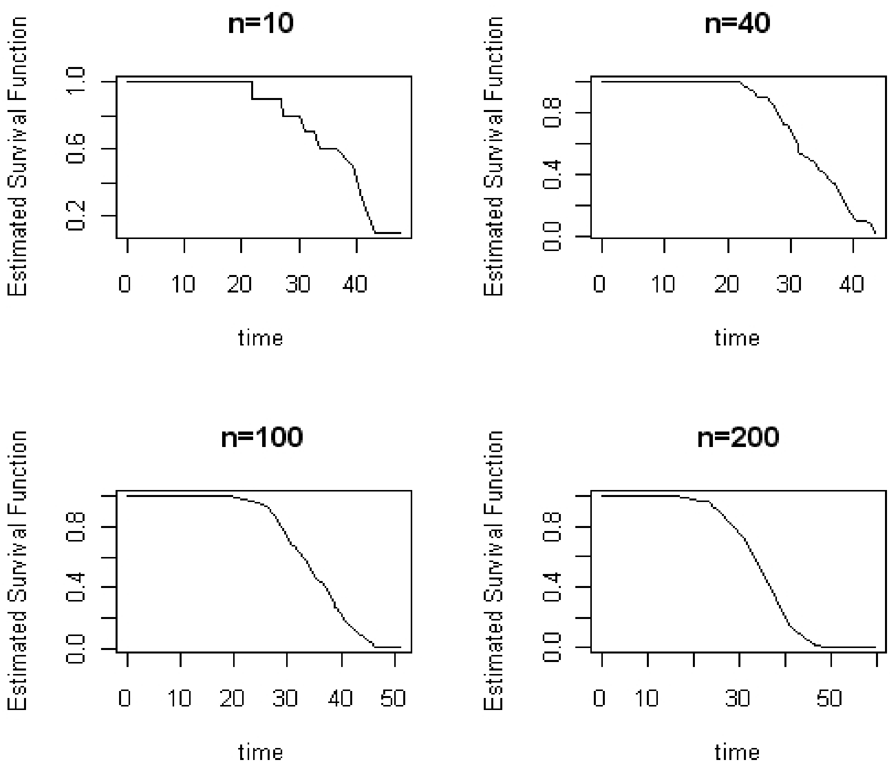

Fig. 6 Comparison of survival function with different sample sizes

6 Simulation on the case of noise model

The actual degradation process can be categorized in two ways: 1-with noise, 2- with measurement error. Here we use the first model which is the one with noise.

Observed degradation value is:

$$Z(T) = Z_r(t)U(t)$$

where
$\ln U(t) = V(t) = \sigma\, W(c(t))$

In this model, W is standard Winer process and independent from A. Pay attention that C is a continuous and increasing function of time. If $\sigma = 0$, the actual degradation process is equal to observed degradation. $Z_r(t)$ is the actual degradation that can be different from the observed degradation because of the noises that have unknown sources. Suppose the actual degradation process is linear; therefore, by $Z_r(t) = \dfrac{t}{A}$. In addition, the simulation of the path of the item has linear model. With supposition of previous part we produce Z (t) by the calculated; W, T, A.
$Z(t) = Z_r(t)\, e^{\sigma W(c(t))}$

In figure 7 and figure 8: The graph of 50 path of the item with different σ is drawn. We can observe when σ is decreasing the graph of the path becomes more smooth.

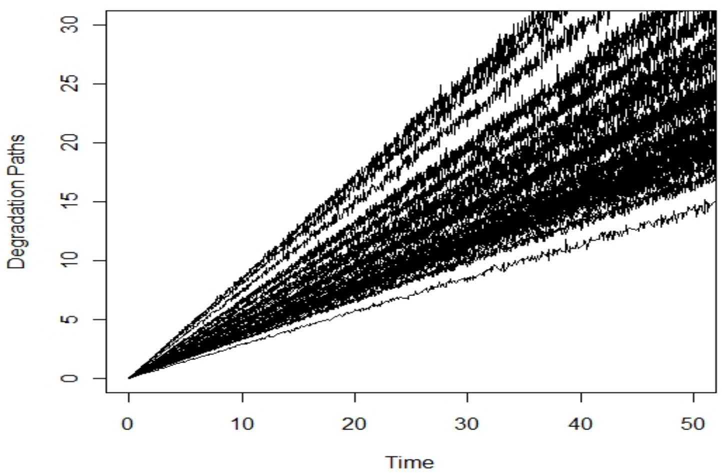

Fig. 7 The graph of the path of item (sigma=0.01)

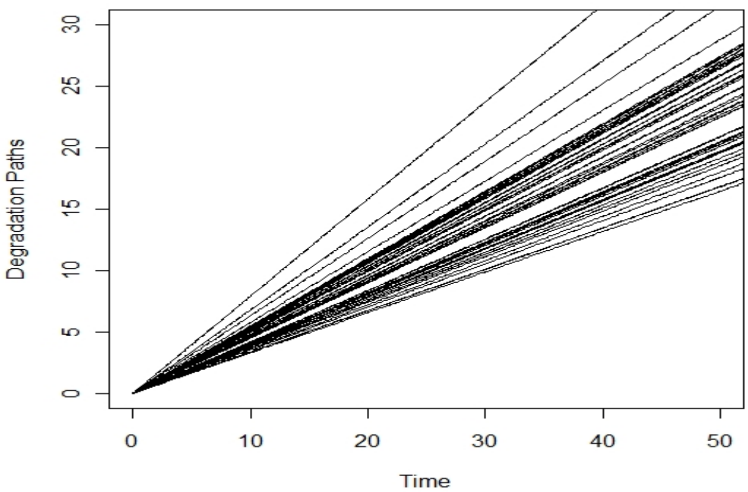

Fig. 8 The graph of path of item (sigma=0.0001)

In figure 9: The graph of survival function is drawn with different σ. We can observe by increasing σ the survival function graph comes close to actual model meaning $\sigma = 0$.

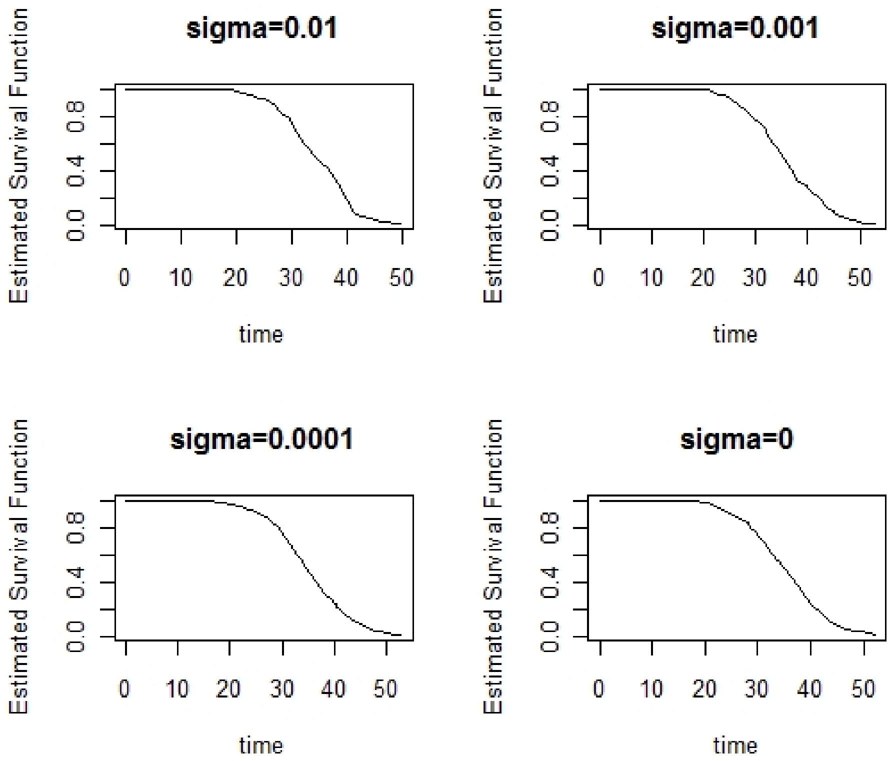

Fig. 9 Comparison of survival function with different σ

7 Conclusions

In this paper, we used simulation in well known example of tire wear rate of buses. We plotted survival function, traumatic and non traumatic failure probability function. Also we used noise in degradation models for plotting degradation path. We will be interested to model degradation with noise in analysis of joint multiple failure mode in future.

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