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Fully Fuzzy Linear Systems

T. Allahviranloo, N. Mikaeilvand*, F. Hoseinzadeh Lotfi, M. Fallah Jelodar

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Abstract As can be seen from the definition of extended operations on fuzzy numbers, subtraction and division of fuzzy numbers are not the inverse operations to addition and multiplication. Hence, to solve the fuzzy equations or a fuzzy system of linear equations analytically, we must use methods without using inverse operators. In this paper, a novel method to find the solutions in which 0 is not the inner point of supports, of fully fuzzy linear systems (shown as **FFLS**) is proposed, if they exist by an analytic approach. The system's parameters were splitted into two groups of nonpositive and nonnegative by solving a multi objective linear programming problem, $MOLP$, and employing an embedding method to transform $n \times n$ **FFLS** to $2n \times 2n$ parametric form linear system and hence, transforming operations on fuzzy numbers to operations on functions. And finally, numerical examples are used to illustrate this approach.

Keywords Fuzzy Numbers, Fully Fuzzy Linear System, Systems of Fuzzy Linear Equations, Embedding Method, Splitting Method.

1 Introduction

System of equations is the simplest and the most useful mathematical model for a lot of problems considered by applied mathematics. In practice, the exact values of coefficients of these systems are not a known rule. This uncertainty may have either probabilistic or non-probabilistic nature. Accordingly, different approaches to the problem and different mathematical tools are needed.

In this article, system of linear equations whose coefficients and right hand sides, and hence solutions, are fuzzy numbers called Fully Fuzzy Linear System **FFLS** are considered.

Abramovich et al., [1], Allahviranloo et al., [2], Buckley and Qu [3-5], Dehghan et al., [6-8], Muzzioli and Reynaerts [9, 10] and Vroman et al., [11-13] suggested different approaches for solving **FFLS**.

F. Abramovich et al. [1] dealt only with LL-type fuzzy numbers (L being arbitrary admissible functions but the same for all coefficients and right-hand sides) where zero does not belong to supports of all coefficients and right-hand sides and uses approximate formulae

*** Corresponding Author.** (✉)
E-mail: mikaeilzand@AOL.com (N. Mikaeilvand)

T. Allahviranloo, F. Hoseinzadeh Lotfi
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.

N. Mikaeilvand
Department of Mathematics, Ardabil Branch, Islamic Azad University, Ardabil, Iran.

M. Fallah Jelodar
Department of Mathematics, Firozkoh Branch, Islamic Azad University, Firozkoh, Iran.

of Dubois and Prade [14] and reduces the problem of approximated solution of LL-type fuzzy linear system to an ordinary (non fuzzy) non-linear optimization problem and solves (approximately) it.

Buckley and Qu [3-5] have discussed the theoretical aspects of the problem in the development of several theorems related to the existence of a solution. They have proposed the following solutions for the **FFLS** like the classical solution X_C, the vector solution X_J, and the marginal solutions X_E and X_I.

In [9, 10] Buckley and Qu's method is extended to a more general fuzzy system of equations $A_1X + b_1 = A_2X + b_2$, with A_1, A_2, b_1 and b_2 fuzzy matrices of fuzzy numbers. The classical solution looks for the fuzzy numbers X to plug into the system yielding exact equality between the fuzzy vectors X and b. Although the classical solution X_C of the system $A_1X + b_1 = A_2X + b_2$ is not equal to the classical solution of the system $AX = b$, if $A = A_1 - A_2$ is nonsingular, their vector solutions X_J are the same (see Theorem 2 of [9]). Thus, Muzzioli and Reynaerts have transformed the system $A_1X + b_1 = A_2X + b_2$ into the **FFLS**, $AX = b$ where $A = A_1 - A_2$ and $b = b_1 - b_2$. Then, they introduced an algorithm to find the vector solution X_J. They also have offered the solution of the fuzzy linear system by means of a nonlinear programming method [10].

Dehghan et al. [6-8] studied finite methods for approximately solving **FFLS**. They represented fuzzy numbers in LR form which are defined and used by Dubois and Prade [14], and they have applied approximately operators between fuzzy numbers in this form and found approximated triangular positive fuzzy number solutions of **FFLS**. Hence, procedures for calculating the solutions of **FFLS** transformed to calculating the solutions of three crisp systems.

Vroman et al. [11-13] suggested two methods to solve **FFLS**. In [11] they proposed a method to solve approximately **FFLS**, and then they proved that their solution is better than Buckley and Q's approximated solution vector X_B. Furthermore, in [12, 13] they proposed an algorithm and improved it by parametric functions.

Allahviranloo et al. [2] dealt with fully fuzzy linear system that the coefficients are positive fuzzy numbers and suggested an analytic approach for finding its solutions which 0 is not inner point of its support.

In this paper, we are going to find solutions of **FFLS** where 0 is not the inner point of its support (they are called non-zero, in this paper). For this reason, we split variables into two groups: nonpositives and nonnegatives and transform operations on fuzzy numbers to operations on functions. We use embedding approach to replace the original $n \times n$ **FFLS** by a $2n \times 2n$ parametric linear system and design an analytic method for calculating the solutions.

The structure of this paper is organized as follows:

In section 2, we discuss some basic definitions, results on fuzzy numbers and **FFLS**. In section 3, we discuss our numerical procedure for finding non-zero solutions of **FFLS** and the proposed algorithm is illustrated by solving some numerical examples. Conclusions are drawn in section 4.

2 Preliminaries

The set of all fuzzy numbers is denoted by $\mathbf{E}$ and defined as follows:

Definition 1. [15] A fuzzy number $\tilde{u}$ is a pair $(\underline{u}(r), \overline{u}(r))$ of functions $\underline{u}(r), \overline{u}(r); 0 \le r \le 1$ which satisfy the following requirements:

- $\underline{u}(r)$ is a bounded monotonic increasing left continuous function;
- $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function;
- $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$.

A crisp number k is simply represented by $\overline{k}(r) = \underline{k}(r) = k; \quad 0 \le r \le 1$ and called singleton.

A fuzzy number $\tilde{a}$ can be represented by its λ-cuts $(0 < \lambda \le 1)$:

$$\tilde{a}^{\lambda} = \{x \mid x \in \mathbb{R}, \tilde{a}(x) \ge \lambda\}$$

and

$$supp\,\tilde{a} = \tilde{a}^{0} = Cl(\{x \mid x \in \mathbb{R}, \tilde{a}(x) > 0\}) = [\underline{a}(0), \overline{a}(0)].$$

For fuzzy number $\tilde{u} = (\underline{u}(r), \overline{u}(r)),\ 0 \le r \le 1$, we will write $(1)\tilde{u} > 0$ if $\underline{u}(0) > 0, (2)\tilde{u} \ge 0$ if $\underline{u}(0) \ge 0, (3)\tilde{u} < 0$ if $\overline{u}(0) < 0, (4)\tilde{u} \le 0$ if $\overline{u}(0) \le 0$.

A fuzzy number is called non-zero fuzzy number, if 0 is not inner point of its support. Based on this definition $\tilde{u}$ is non-zero fuzzy number if and only if $\tilde{u} \le 0$ or $\tilde{u} \ge 0$ [1].

For arbitrary $\tilde{u} = (\underline{u}(r), \overline{u}(r))$ and $\tilde{v} = (\underline{v}(r), \overline{v}(r))$, addition $(\tilde{u} + \tilde{v})$, subtraction $(\tilde{u} - \tilde{v})$ and multiplication $(\tilde{u}.\tilde{v})$ are defined as:

Addition:

$$(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r), \quad (\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r), \tag{1}$$

Subtraction:

$$(\underline{u-v})(r) = \underline{u}(r) - \overline{v}(r), \quad (\overline{u-v})(r) = \overline{u}(r) - \underline{v}(r), \tag{2}$$

Multiplication:

$$\begin{aligned} (\underline{uv})(r) &= min\{\overline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \underline{u}(r)\underline{v}(r)\}, \\ (\overline{uv})(r) &= max\{\overline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \underline{u}(r)\underline{v}(r)\}. \end{aligned} \tag{3}$$

Many problems in various areas of science can be solved by solving system of linear equations. If system's parameters are vague or imprecise, this uncertainty can be represented by fuzzy numbers rather than crisp numbers, and the system of linear equations is called fuzzy system of linear equations.

Fuzzy system of linear equations whose coefficients and right hand sides are fuzzy numbers is called Fully Fuzzy Linear System **FFLS**. A fuzzy solution of **FFLS** is defined

[1] except singleton 0

as:

Definition 2. The Vector of fuzzy numbers $(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n)^t$ given by $\tilde{x}_i = (\underline{x_i}(r), \overline{x_i}(r)),\ 1 \le i \le n,\ 0 \le r \le 1$ is called a fuzzy solution of **FFLS** , if

$$\underline{\sum_{j=1}^{n} a_{ij}x_j(r)} = \sum_{j=1}^{n} \underline{a_{ij}x_j}(r) = \underline{b_i}(r), \quad \overline{\sum_{j=1}^{n} a_{ij}x_j}(r) = \sum_{j=1}^{n} \overline{a_{ij}x_j}(r) = \overline{b_i}(r), \quad i = 1, \cdots, n. \tag{4}$$

The **FFLS** may not have fuzzy solution. Unfortunately, we do not know, where **FFLS** has a fuzzy solution

The non-zero fuzzy solution of **FFLS** is defined as follows:

Definition 3. A fuzzy solution $(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n)^t$ of **FFLS** is called non-zero fuzzy solution, if for all $i\ (i = 1,2,...,n),\ \tilde{x}_i$ is non-zero fuzzy number.

Necessary and sufficient condition for the existence of a non-zero fuzzy solution of **FFLS** is:

Theorem 1. [2] If **FFLS** $\mathbf{AX = b}$ has a fuzzy solution, then $\mathbf{AX = b}$ has non-zero fuzzy solution if and only if $0-$cut system of linear system represented by $\mathbf{A^0X^0 = b^0}$ has non-zero solution.

3 The Proposed Method

In this section, we are going to find non-zero fuzzy solutions of **FFLS** . We suppose that $\tilde{a}_{ij}$ the elements of A, are in three forms of: 1- $\tilde{a}_{ij} \ge 0$ 2- $\tilde{a}_{ij} \le 0$ 3- $\underline{a_{ij}}(r) \le 0\ and\ \overline{a_{ij}}(r) \ge 0$ and the proposed method can not solve other systems in which the coefficients are not in three forms.

Let $\mathbf{AX = b}$ be **FFLS** . Consider i^{th} equation of this system:

$$\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j = \tilde{b}_i \quad ,i = 1,...,n. \tag{5}$$

Since we suppose coefficients are in three forms, three $n \times n$ matrices of fuzzy numbers A_1, A_2, A_3 are defined as follows:

$$(A_1)_{ij} = \begin{cases} \tilde{a}_{ij}, & if\ \tilde{a}_{ij} \ge 0, \\ 0, & Otherwise, \end{cases} \qquad (A_2)_{ij} = \begin{cases} \tilde{a}_{ij}, & if\ \tilde{a}_{ij} \le 0, \\ 0, & Otherwise, \end{cases}$$

$$(A_3)_{ij} = \begin{cases} \tilde{a}_{ij}, & if\ \underline{a_{ij}}(r) \le 0\ and\ \overline{a_{ij}}(r) \ge 0, \\ 0, & Otherwise. \end{cases} \tag{6}$$

It is clear that

$$A = A_1 + A_2 + A_3, \tag{7}$$

and

$$AX = A_1X + A_2X + A_3X. \tag{8}$$

Hence the i^{th} equation of system $\mathbf{AX = b}$ is transformed to this equation:

$$\tilde{b}_i = \sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j = \sum_{j=1}^{n} \tilde{a}_{1ij}\tilde{x}_j + \sum_{j=1}^{n} \tilde{a}_{2ij}\tilde{x}_j + \sum_{j=1}^{n} \tilde{a}_{3ij}\tilde{x}_j, i = 1,...,n. \tag{9}$$

Now, let $\mathbf{AX = b}$ has a non-zero fuzzy solution, we define

$$J = \{j \mid 1 \le j \le n \ , \tilde{x}_j \ge 0\}. \tag{10}$$

Hence (9) can be rewritten as:

$$\tilde{b}_i = \sum_{j\in J} \tilde{a}_{1ij}\tilde{x}_j + \sum_{j\in J} \tilde{a}_{2ij}\tilde{x}_j + \sum_{j\in J} \tilde{a}_{3ij}\tilde{x}_j + \sum_{j\notin J} \tilde{a}_{1ij}\tilde{x}_j + \sum_{j\notin J} \tilde{a}_{2ij}\tilde{x}_j + \sum_{j\notin J} \tilde{a}_{3ij}\tilde{x}_j, i = 1,...,n. \tag{11}$$

If we define two n-vector $\tilde{Y} = (\tilde{y}_1, \tilde{y}_2,...,\tilde{y}_n)^t$ and $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2,...,\tilde{z}_n)^t$ where

$$\tilde{z}_j = \begin{cases} \tilde{x}_j & \text{if } j \in J, \\ 0 & \text{if } j \notin J, \end{cases} \quad \tilde{y}_j = \begin{cases} \tilde{x}_j & \text{if } j \notin J, \\ 0 & \text{if } j \in J. \end{cases} \tag{12}$$

It is obvious that

$$\tilde{z}_j + \tilde{y}_j = \tilde{x}_j, \ \ 1 \le j \le n. \tag{13}$$

If we replace $\tilde{x}_j$ in (13) with $\tilde{z}_j + \tilde{y}_j$, (11) can be rewritten as follows:

$$\begin{aligned}\tilde{b}_i = \sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j &= \sum_{j=1}^{n} \tilde{a}_{1ij}\tilde{x}_j + \sum_{j=1}^{n} \tilde{a}_{2ij}\tilde{x}_j + \sum_{j=1}^{n} \tilde{a}_{3ij}\tilde{x}_j \\ &= \sum_{j=1}^{n} \tilde{a}_{1ij}\tilde{z}_j + \sum_{j=1}^{n} \tilde{a}_{1ij}\tilde{y}_j + \sum_{j=1}^{n} \tilde{a}_{2ij}\tilde{z}_j \\ &+ \sum_{j=1}^{n} \tilde{a}_{2ij}\tilde{y}_j + \sum_{j=1}^{n} \tilde{a}_{3ij}\tilde{z}_j + \sum_{j=1}^{n} \tilde{a}_{3ij}\tilde{y}_j, i = 1,...,n.\end{aligned} \tag{14}$$

If we use the definition 2, and if $\tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_n)^t$ is a fuzzy number solution of $\mathbf{AX = b}$, the following equations must be true:

$$\begin{aligned}\underline{b}_i(r) &= \sum_{j=1}^{n} \underline{a_{1ij}z_j}(r) + \sum_{j=1}^{n} \underline{a_{1ij}y_j}(r) + \sum_{j=1}^{n} \underline{a_{2ij}z_j}(r) \\ &+ \sum_{j=1}^{n} \underline{a_{2ij}y_j}(r) + \sum_{j=1}^{n} \underline{a_{3ij}z_j}(r) + \sum_{j=1}^{n} \underline{a_{3ij}y_j}(r), \quad i = 1,...,n,\end{aligned} \tag{15}$$

and

$$\begin{aligned}\overline{b}_i(r) &= \sum_{j=1}^{n}\overline{a_{1ij}z_j}(r)+\sum_{j=1}^{n}\overline{a_{1ij}y_j}(r)+\sum_{j=1}^{n}\overline{a_{2ij}z_j}(r)\\ &+\sum_{j=1}^{n}\overline{a_{2ij}y_j}(r)+\sum_{j=1}^{n}\overline{a_{3ij}z_j}(r)+\sum_{j=1}^{n}\overline{a_{3ij}y_j}(r),\quad i=1,\ldots,n.\end{aligned} \tag{16}$$

To split $\widetilde{X}$ in two groups of *non positive* and *non negative*, we must answer this question: Is $\widetilde{x}_j$ non negative fuzzy number? In order to answer to this question, we define w_j and $\widetilde{v}_j$ as:

$$w_j = min\{x \in \mathrm{R} \mid x \geq 0,\ x+\widetilde{x}_j \geq 0\}. \tag{17}$$

And

$$\widetilde{v}_j = \widetilde{x}_j + w_j \geq 0,\ \ 1 \leq j \leq n. \tag{18}$$

Since w_j, $(1 \leq j \leq n)$ is singleton, we can write:

$$\widetilde{x}_j = \widetilde{v}_j - w_j\ \ ,\ \ 1 \leq j \leq n. \tag{19}$$

$w_j = 0$ if and only if $\widetilde{x}_j$ is nonnegative fuzzy number and hence $w_j = 0$ if and only if $j \in J$. Hence (15), (16) can be rewritten as:

$$\begin{aligned}\underline{b}_i(r) &= \sum_{j=1}^{n}\underline{a_{1ij}z_j}(r)+\sum_{j=1}^{n}\underline{a_{1ij}(v_j-w_j)}(r)+\sum_{j=1}^{n}\underline{a_{2ij}z_j}(r)\\ &+\sum_{j=1}^{n}\underline{a_{2ij}(v_j-w_j)}(r)+\sum_{j=1}^{n}\underline{a_{3ij}z_j}(r)+\sum_{j=1}^{n}\underline{a_{3ij}(v_j-w_j)}(r),\quad i=1,\ldots,n,\end{aligned} \tag{20}$$

and

$$\begin{aligned}\overline{b}_i(r) &= \sum_{j=1}^{n}\overline{a_{1ij}z_j}(r)+\sum_{j=1}^{n}\overline{a_{1ij}(v_j-w_j)}(r)+\sum_{j=1}^{n}\overline{a_{2ij}z_j}(r)\\ &+\sum_{j=1}^{n}\overline{a_{2ij}(v_j-w_j)}(r)+\sum_{j=1}^{n}\overline{a_{3ij}z_j}(r)+\sum_{j=1}^{n}\overline{a_{3ij}(v_j-w_j)}(r),\quad i=1,\ldots,n.\end{aligned} \tag{21}$$

Since $\widetilde{z}_j \geq 0,\ \ \widetilde{v}_j - w_j \leq 0,\ \ (1 \leq j \leq n)$ and by applying (19) we can write:

$$
\begin{aligned}
&\underline{a_{1ij} z_j}(r) = \underline{a_{1ij}}(r).\underline{z_j}(r), && \overline{a_{1ij} z_j}(r) = \overline{a_{1ij}}(r).\overline{z_j}(r),\\
&\underline{a_{1ij}(v_j - w_j)}(r) = \overline{a_{1ij}}(r).\underline{v_j}(r) - \overline{a_{1ij}}(r) w_j, && \overline{a_{1ij}(v_j - w_j)}(r) = \underline{a_{1ij}}(r).\overline{v_j}(r) - \underline{a_{1ij}}(r) w_j,\\
&\underline{a_{2ij} z_j}(r) = \underline{a_{2ij}}(r).\overline{z_j}(r), && \overline{a_{2ij} z_j}(r) = \overline{a_{2ij}}(r).\underline{z_j}(r),\\
&\underline{a_{2ij}(v_j - w_j)}(r) = \overline{a_{2ij}}(r).\overline{v_j}(r) - \overline{a_{2ij}}(r) w_j, && \overline{a_{2ij}(v_j - w_j)}(r) = \underline{a_{2ij}}(r).\underline{v_j}(r) - \underline{a_{2ij}}(r) w_j,\\
&\underline{a_{3ij} z_j}(r) = \underline{a_{3ij}}(r).\overline{z_j}(r), && \overline{a_{3ij} z_j}(r) = \overline{a_{3ij}}(r).\overline{z_j}(r),\\
&\underline{a_{3ij}(v_j - w_j)}(r) = \overline{a_{3ij}}(r).\underline{v_j}(r) - \overline{a_{3ij}}(r) w_j, && \overline{a_{3ij}(v_j - w_j)}(r) = \underline{a_{3ij}}(r).\underline{v_j}(r) - \underline{a_{3ij}}(r) w_j,
\end{aligned}
\tag{22}
$$

Now, if we replace the above expressions in (20) and (21) ,they can be rewritten as:

$$
\begin{aligned}
&\sum_{j=1}^{n} \underline{a_{1ij}}(r).\underline{z_j}(r) + \sum_{j=1}^{n}(\underline{a_{2ij}}(r) + \underline{a_{3ij}}(r))\overline{z_j}(r) + \sum_{j=1}^{n}(\overline{a_{1ij}}(r) + \overline{a_{3ij}}(r)).\underline{v_j}(r) + \sum_{j=1}^{n}\overline{a_{2ij}}(r).\overline{v_j}(r)\\
&\qquad = \underline{b_i}(r) + \sum_{j=1}^{n}(\overline{a_{1ij}}(r) + \overline{a_{2ij}}(r) + \overline{a_{3ij}}(r))w_j, \qquad i = 1,\dots,n,
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^{n} \overline{a_{2ij}}(r).\underline{z_j}(r) + \sum_{j=1}^{n}(\overline{a_{1ij}}(r) + \overline{a_{3ij}}(r)).\overline{z_j} + \sum_{j=1}^{n}(\underline{a_{2ij}}(r) + \underline{a_{3ij}}(r)).\underline{v_j} + \sum_{j=1}^{n}\underline{a_{1ij}}(r).\overline{v_j}\\
&\qquad = \overline{b_i}(r) + \sum_{j=1}^{n}(\underline{a_{1ij}}(r) + \underline{a_{2ij}}(r) + \underline{a_{3ij}}(r))w_j, \qquad i = 1,\dots,n.
\end{aligned}
\tag{24}
$$

Hence i^{th} equation of $AX = b$ is transformed to two parametric equations (23) and (24). Now we illustrate these equations in matrix forms. If C_l, $l = 1,2,3,4,$ are parametric $n \times n$ matrices by elements

$$
\begin{aligned}
&(C_1)_{ij} = \underline{a_{1ij}}(r), && (C_2)_{ij} = \underline{a_{2ij}}(r) + \underline{a_{3ij}}(r),\\
&(C_3)_{ij} = \overline{a_{1ij}}(r) + \overline{a_{3ij}}(r), && (C_4)_{ij} = \overline{a_{2ij}}(r),
\end{aligned}
\tag{25}
$$

and if Z_1, Z_2, V_1, V_2, B_1, B_2 and W are parametric n-vectors by elements

$$
\begin{aligned}
&Z_1 = (\underline{z_1}(r),\dots,\underline{z_n}(r))^t, \qquad Z_2 = (\overline{z_1}(r),\dots,\overline{z_n}(r))^t,\\
&V_1 = (\underline{v_1}(r),\dots,\underline{v_n}(r))^t, \qquad V_2 = (\overline{v_1}(r),\dots,\overline{v_n}(r))^t,\\
&B_1 = (\underline{b_1}(r),\dots,\underline{b_n}(r))^t, \qquad B_2 = (\overline{b_1}(r),\dots,\overline{b_n}(r))^t,\\
&W = (\underline{w_1}(r),\dots,\underline{w_n}(r))^t.
\end{aligned} \tag{26}
$$

the **FFLS** $\mathbf{AX = b}$ can be represented in matrix form as:

$$
\begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ C_4 & C_3 & C_2 & C_1 \end{pmatrix}
\begin{pmatrix} Z_1 \\ Z_2 \\ V_1 \\ V_2 \end{pmatrix}
=
\begin{pmatrix} B_1 + (C_3 + C_4)W \\ B_2 + (C_1 + C_2)W \end{pmatrix} \tag{27}
$$

where this coefficients matrix is represented in $2n \times 4n$. But in fact, by definitions of Z_1, Z_2, V_1 and V_2, $2n$ elements of variable vector are zero and hence $2n$ columns of coefficients matrix are omitted and hence we replace $n \times n$ **FFLS** by an $2n \times 2n$ system of linear parametric equations. But to split $\widetilde{X}$, we want to have information about nonpositivity or nonnegativity of $\widetilde{x}_j$ and hence we must determine w_j for all $(1 \le \mathrm{j} \le \mathrm{n})$ simultaneously, where the problem of determining of w_j for all $(1 \le j \le n)$, are transformed to following $MOLP$:

$$
\begin{aligned}
&\text{Min } w_1\\
&\text{Min } w_2\\
&\quad\vdots\\
&\text{Min } w_n\\
&\text{s.t.}\\
&\sum_{j=1}^{n} \underline{a_{1ij}}(0).\underline{z_j}(0)+\sum_{j=1}^{n}(\underline{a_{2ij}}(0)+\underline{a_{3ij}}(0))\overline{z_j}(0)+\sum_{j=1}^{n}(\overline{a_{1ij}}(0)+\overline{a_{3ij}}(0)).\underline{v_j}(0)+\sum_{j=1}^{n}\overline{a_{2ij}}(0).\overline{v_j}(0)\\
&\qquad\qquad =\underline{b_i}(0)+\sum_{j=1}^{n}(\overline{a_{1ij}}(0)+\overline{a_{2ij}}(0)+\overline{a_{3ij}}(0))w_j,\ i=1,\dots,n,\\
&\sum_{j=1}^{n} \overline{a_{2ij}}(0).\underline{z_j}(0)+\sum_{j=1}^{n}(\overline{a_{1ij}}(0)+\overline{a_{3ij}}(0)).\overline{z_j}(0)+\sum_{j=1}^{n}(\underline{a_{2ij}}(0)+\underline{a_{3ij}}(0)).\underline{v_j}(0)+\sum_{j=1}^{n}\underline{a_{1ij}}(0).\overline{v_j}(0)\\
&\qquad\qquad =\overline{b_i}(0)+\sum_{j=1}^{n}(\underline{a_{1ij}}(0)+\underline{a_{2ij}}(0)+\underline{a_{3ij}}(0))w_j,\ i=1,\dots,n,\\
&\underline{z_j}(0)\le\overline{z_j}(0),\quad j=1,\dots,n,\\
&\underline{v_j}(0)\le\overline{v_j}(0),\quad j=1,\dots,n,\\
&\underline{z_j}(0)\ge 0,\quad j=1,\dots,n,\\
&\underline{v_j}(0)\ge 0,\quad j=1,\dots,n,\\
&w_j\ge 0,\quad j=1,\dots,n.
\end{aligned}
\tag{28}
$$

If this *MOLP* does not have any feasible solution, **FFLS** is unsolvable with the proposed method because, its constraint is 0-cut of **FFLS** . If the above MOLP has a feasible solution, we can find $\tilde{z}_j\ (j=1,\dots,n)$ and hence the 2n columns of coefficient matrix in (27) are omitted and this system is transformed to $2n\times 2n$ parametric linear system. Maybe, this *MOLP* has alternative solutions, but we only require a solution which satisfy the following condition: if $w_j>0$ then $\tilde{z}_j=0\,(j=1,\dots,n)$. After solving the above *MOLP* and finding the required solution, and replacing it in (18) and solving this system, if its solutions define non-zero fuzzy numbers, **FFLS** $\mathbf{AX}=\mathbf{b}$ will have non-zero fuzzy number solutions.

The algorithm for solving (**FFLS**) and finding its non-zero solution is illustrated as follows:

The Alghorithm Of Non – Zero Solution of (**FFLS**) ′s

Suppose $\mathbf{AX}=\mathbf{b}$ is a (**FFLS**).

1. Solve MOLP (28) and find its solution. If it has feasible solutions where if $w_j>0$ then $\tilde{z}_j=0,(j=1,\dots,n)$, this system has non-zero solution then go to 2 or else go to 7.
2. Define J and hence Z and Y.
3. Transform $\mathbf{AX}=\mathbf{b}$ to (27) system using the solutions of MOLP (28).
4. Omit the $2n$ columns of matrix of coefficient respect to zero elements of Z and V .
5. Solve $2n\times 2n$ parametric system (27).

6. If the solutions of (27) can define fuzzy numbers, these solutions are non-zero fuzzy number solution of (**FFLS**) $\mathbf{AX}=\mathbf{b}$ and go to 8.
7. This system has a zero solution or this system does not have any fuzzy number solution and this algorithm can not solve it.
8. End.

Now, we illustrate our method using numerical examples.

Example 1. Consider the system of equations

$$\begin{cases} (4+r,6-r)x_1+(5+r,8-2r)x_2=(40+10r,67-17r) \\ \\ (6+r,7)x_1+\ (4,5-r)x_2 \quad =(43+5r,55-7r) \\ \end{cases}$$

Dehghan et al. in [6] solved this system approximately. Their solutions are $x_1=(\frac{43}{11}+\frac{1}{11}r,4)$ and $x_2=(\frac{54}{11}+\frac{1}{11}r,\frac{21}{4}-\frac{1}{4}r),$ that satisfy only in 1-cut.

We solve this system by our algorithm as follows:

First, we solve the following MOLP:

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The solution of this system is $w_1=w_2=0$ and hence $v_1=v_2=0$ and 2×2 matrix of coefficients can be replaced by 4×4 parametric coefficient matrix as follows:

$C_l,\ l=1,\dots,4, B_1$ and B_2 are defined as:

$$C_1=\begin{pmatrix} 4+r & 5+r \\ 6+r & 4 \\ \end{pmatrix},\quad C_3=\begin{pmatrix} 6-r & 8-2r \\ 7 & 5-r \\ \end{pmatrix},$$

$$B_1=\begin{pmatrix} 40+10r \\ 43+5r \\ \end{pmatrix},\qquad B_2=\begin{pmatrix} 67-17r \\ 55-7r \\ \end{pmatrix},$$

$$C_2=C_4=\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \end{pmatrix}.$$

and

$$\begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ C_4 & C_3 & C_2 & C_1 \\ \end{pmatrix}=\begin{pmatrix} 4+r & 5+r & 0 & 0 & 6-r & 8-2r & 0 & 0 \\ 6+r & 4 & 0 & 0 & 7 & 5-r & 0 & 0 \\ 0 & 0 & 6-r & 8-2r & 0 & 0 & 4+r & 5+r \\ 0 & 0 & 7 & 5-r & 0 & 0 & 6+r & 4 \\ \end{pmatrix}.$$

Since $v_1=v_2=0,$ 4 columns of the coefficient matrix are omitted and the coefficient matrix is transformed to

$$\begin{pmatrix} 4+r & 5+r & 0 & 0 \\ 6+r & 4 & 0 & 0 \\ 0 & 0 & 6-r & 8-2r \\ 0 & 0 & 7 & 5-r \end{pmatrix}.$$

Hence, 2×2 **FFLS**, $\mathbf{AX}=\mathbf{b}$ is transformed to following 4×4 parametric system:

$$\begin{pmatrix} 4+\mathrm{r} & 5+\mathrm{r} & 0 & 0 \\ 6+\mathrm{r} & 4 & 0 & 0 \\ 0 & 0 & 6-\mathrm{r} & 8-2\mathrm{r} \\ 0 & 0 & 7 & 5-\mathrm{r} \end{pmatrix}\begin{pmatrix} \underline{z}_1 \\ \underline{z}_2 \\ \overline{z}_1 \\ \overline{z}_2 \end{pmatrix} =$$

$$\begin{pmatrix} 40+10\mathrm{r}+(6-\mathrm{r})\mathrm{w}_1+(8-2\mathrm{r})\mathrm{w}_2 \\ 43+5\mathrm{r}+7\mathrm{w}_1+(5-\mathrm{r})\mathrm{w}_2 \\ 67-17\mathrm{r}+(4+\mathrm{r})\mathrm{w}_1+(5+\mathrm{r})\mathrm{w}_2 \\ 55-7\mathrm{r}+(6+\mathrm{r})\mathrm{w}_1+4\mathrm{w}_2 \end{pmatrix} = \begin{pmatrix} 40+10\mathrm{r} \\ 43+5\mathrm{r} \\ 67-17\mathrm{r} \\ 55-7\mathrm{r} \end{pmatrix}.$$

and its solutions are:

$$\underline{z}_1(r)=\underline{x}_1(r)=\frac{5r^2+28r+55}{r^2+7r+14},\quad \overline{z}_1(r)=\overline{x}_1(r)=\frac{3r^2+14r-105}{r^2+3r-26},$$

$$\underline{z}_2(r)=\underline{x}_2(r)=\frac{5r^2+37r+68}{r^2+7r+14},\quad \overline{z}_2(r)=\overline{x}_2(r)=\frac{7r^2+22r-139}{r^2+3r-26}.$$

Since

$\underline{x}_1(r)$ and $\underline{x}_2(r)$ are bounded monotonic increasing left continuous functions.
$\overline{x}_1(r)$ and $\overline{x}_2(r)$ are bounded monotonic decreasing left continuous functions.
$\underline{x}_1(r)\le \overline{x}_1(r), (0\le r\le 1)$ and $\underline{x}_2(r)\le \overline{x}_2(r), (0\le r\le 1)$

Parametric system solutions define fuzzy numbers and non-zero fuzzy number solutions of this **FFLS** are:

$$x_1=\left(\frac{5r^2+28r+55}{r^2+7r+14},\frac{3r^2+14r-105}{r^2+3r-26}\right) \text{ and } x_2=\left(\frac{5r^2+37r+68}{r^2+7r+14},\frac{7r^2+22r-139}{r^2+3r-26}\right).$$

Example 2. Consider the fully fuzzy linear system $\mathbf{AX}=\mathbf{b}$ in which

$$\mathbf{A}=\begin{pmatrix} (-5+r,-1-r) & (-3+r^2,-2r^2) \\ (2r,5-3r) & (1+3r^2,6-2r) \end{pmatrix} \text{ and } \mathbf{b}=\begin{pmatrix} (-r^3+7r^2+5r-15,-2r^3-14r+20) \\ (3r^3-3r^2+23r-19,-2r^2-16r+30) \end{pmatrix}.$$

We solve this system by our method as follows:
First, we solve the following MOLP:

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The solution of this system is $w_1 = 4$ and $w_2 = 0$ and hence $v_1 = z_2 = 0$ and 2×2 matrix of coefficients can be replaced by 4×4 parametric matrix of coefficients.

Hence, $J = \{1\}$ and hence we must define $z_1 = x_1, z_2 = 0, y_1 = 0, y_2 = x_2$. C_l, $l = 1,2,3,4$, B_1 and B_2 are defined as:

$$C_1 = \begin{pmatrix} 0 & 0 \\ 2r & 1+3r^2 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} -5+r & -3+r^2 \\ 0 & 0 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & 0 \\ 5-3r & 6-2r \end{pmatrix} \qquad C_4 = \begin{pmatrix} -1-r & -2r^2 \\ 0 & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} -r^3+7r^2+5r-15 \\ 3r^3-3r^2+23r-19 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -2r^3-14r+20r \\ -2r^3-16r+30 \end{pmatrix}.$$

and

$$\begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ C_4 & C_3 & C_2 & C_1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & -5+r & -3+r^2 & 0 & 0 & -1-r & -2r^2 \\ 2r & 1+3r^2 & 0 & 0 & 5-3r & 6-2r & 0 & 0 \\ -1-r & -2r^2 & 0 & 0 & -5+r & -3+r^2 & 0 & 0 \\ 0 & 0 & 5-3r & 6-2r & 0 & 0 & 2r & 1+3r^2 \end{pmatrix}.$$

Since $v_1 = z_2 = 0$, 4 columns of matrix of coefficients are omitted and matrix of coefficients is transformed to

$$\begin{pmatrix} 0 & -3+r^2 & 0 & -2r^2 \\ 2r & 0 & 5-3r & 0 \\ -1-r & 0 & -5+r & 0 \\ 0 & 6-2r & 0 & 1+3r^2 \end{pmatrix}.$$

Hence, 2×2 **FFLS**, $\mathbf{AX} = \mathbf{b}$ is transformed to following 4×4 parametric system:

$$\begin{pmatrix} 0 & -3+r^2 & 0 & -2r^2 \\ 2r & 0 & 5-3r & 0 \\ -1-r & 0 & -5+r & 0 \\ 0 & 6-2r & 0 & 1+3r^2 \end{pmatrix}\begin{pmatrix} \underline{z_1} \\ \overline{z_1} \\ \underline{v_2} \\ \overline{v_2} \end{pmatrix} =$$

$$\begin{pmatrix} -r^3+7r^2+5r-155+(-1-r)w_1+(-2r^3)w_2 \\ 3r^3-3r^2+23r-19+(5-3r)w_1+(6-2r)w_2 \\ -2r^3-14r+20+(-5+r)w_1+(-3+r^2)w_2 \\ -2r^2-16r+30+2rw_1+(1+3r^2)w_2 \end{pmatrix} =$$

$$\begin{pmatrix} -r^3+7r^2+5r-155+(-1-r)4 \\ 3r^3-3r^2+23r-19+(5-3r)4 \\ -2r^3-14r+20+(-5+r)4 \\ -2r^2-16r+30+2r(4) \end{pmatrix} = \begin{pmatrix} -r^3+7r^2+r-159 \\ 3r^3-3r^2+11r+1 \\ -2r^3-10r \\ -2r^2-8r+30 \end{pmatrix}.$$

and its solutions are:

$$\underline{v}_1(r)-4=\underline{x}_1(r)=2r-4, \quad \overline{v}_1(r)-4=\overline{x}_1(r)=-2r,$$
$$\underline{z}_2(r)=\underline{x}_2(r)=1+r, \quad \overline{z}_2(r)=\overline{x}_2(r)=5-r.$$

since

$\underline{x}_1(r)$ and $\underline{x}_2(r)$ are bounded monotonic increasing left continuous functions.

$\overline{x}_1(r)$ and $\overline{x}_2(r)$ are bounded monotonic decreasing left continuous functions.

$\underline{x}_1(r) \le \overline{x}_1(r), (0 \le r \le 1)$ and $\underline{x}_2(r) \le \overline{x}_2(r), (0 \le r \le 1)$

Parametric system solutions define fuzzy numbers and non-zero fuzzy number solutions of this **FFLS** are:

$x_1 = (-4+2r,-2r)$, $x_2 = (1+r,5-r)$. Which are the exact solutions.

4 Conclusion

In this paper, we found non-zero solution of fully fuzzy linear system of equations, analytically. For this means an algorithm is introduced. In this Algorithm, $n \times n$ system $\mathbf{A}X = b$ is transformed to $2n \times 4n$ parametric system and then to $2n \times 2n$ parametric system. For this aim, We first solve MOLP (28) and find its solutions. If its solutions are non-zero, we transform $2n \times 4n$ parametric system to $2n \times 2n$ parametric form linear system and solve it. Unfortunately, we do not know, where **FFLS** has a fuzzy solution and hence, in this method, if solutions of (27) can define fuzzy numbers; we can say that the algorithm has found non-zero fuzzy numbers solutions. Note that it may happen that MOLP (28) does not have a solution and hence, we can not solve fully fuzzy linear system of equations by the proposed algorithm. May be MOLP has a solution but $(\mathbf{AX} = \mathbf{b})$ has zero solution and hence, the algorithm can not find its solutions. The proposed algorithm does not have any restriction in parametric form of fuzzy numbers, and in some situations where the system has a non-zero solution, e.g. Ex. 2, the proposed method can find analytically solutions.

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