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Ranking Decision Making Units, using Non-radial Model, applying Bootstrap

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Abstract Data envelopment analysis (DEA) is a mathematical programming method in Operations Research that can be used to distinguish between efficient and inefficient decision making units (DMUs). However, the conventional DEA models do not have the ability to rank the efficient DMUs. This article suggests bootstrapping method for ranking measures of technical efficiency as calculated via non-radial models of DEA and a numerical example is used to illustrate the method.

Keywords: Data Envelopment Analysis, Ranking, Bootstrap, Non-radial Models.

1 Introduction

DEA is a nonparametric linear programming method used for determining the efficiency of a set of companies as compared to the best practice frontier. It can be employed to analyze organizations. The application of the method in the transport sector is wide-spread, especially in the evaluation of airports, ports, railways and urban transport companies [10]. The aim of the present article is ranking decision making units (DMUs) in non-radial models. As is well known, DEA assigns the efficiency value of one to the DMUs which are strongly or weakly efficient. All the DMUs lying on the efficiency frontier are considered efficient and thus there might be several units with an efficiency value of unity. To be able to distinguish the performance of these units, numerous ranking methods have been developed since the introduction of the DEA technique. It must be noted that several of the solutions found in the literature are not anymore distinct measures that can easily be categorized into one or the other group of applications, the approaches frequently overlap. Hence, the aim was to give a clear and concise picture of the models at hand and list them below the heading which is the most revealing as to the content of the method. More recently, bootstrapping method in radial model have been studied by Ebadi and Jahanshahloo [4].

This article shows how bootstrapping techniques can be used to ranking the efficiency scores produced by non-radial model. The bootstrap is a nonparametric approach to statistical inference. Alternatively, parametric or semi-parametric methods could be used to ranking efficient units. The bootstrap was chosen because, like the linear programming approach itself, it is nonparametric and therefore does not impose any structure on the shape of the efficiency distributions. The article proceeds as follows. First, the non-radial model approach to efficiency measurement is outlined. Our implementation of the bootstrap to establish statistical properties of the efficiency measure is then described. We then offer an illustration of this method by applying it to 74 high schools.

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2 Background

DEA provides a measure of the efficiency of a DMU relative to other such units, producing the same outputs with the same inputs. The units to be compared may be enterprisers, banks, schools, hospitals, etc.[3]. DEA is related to the concept of technical efficiency and can be considered as a generalization of efficiency measure.

Assume that there is a sample of n DMUs, each producing an s-dimensional row vector of outputs y, from an m-dimensional row vector of inputs x. Technology governs the transformation of inputs into outputs; the reference technology relative to which efficiency is assessed is given by the input requirement set $L(y)=\{x: \ x \text{ can produce } y\}$. Farrell's [6] input-based measure of technical efficiency for each observation t=1,…,n is given by:

$$TE_t(x_t, y_t) = min\{\theta_t : \theta_t x_t \in L(y_t)\} \tag{1}$$

that is, t^{th} DMU's observed input vector (x_t) is scalar $(0 \le \theta_t \le 1)$ until it is still just able to produce the observed level of output (y_t). The solution, $TE_t = \theta_t^*$, gives the proportion of the t^{th} DMU's actual input vector that is technologically necessary to produce its observed output vector given the best practice technology as revealed by the observed data. The vector $x_t^* = \theta_t^* x_t$ would give the technically efficient (optimal) input vector for the t^{th} DMU.

One way to calculate this measure of technical efficiency is by the following linear programming problem once for each $DMU_t, t=1,...,n$:

$$\begin{aligned} \min \ & \theta_t \\ st: \ & \lambda Y \ge y_t \\ & \lambda . X \le \theta . x_t \\ & e\lambda = 1 \\ & \lambda \ge 0 \end{aligned} \tag{2}$$

Where Y is the n by s matrix of the observed outputs of all DMUs, X is the n by m matrix of the observed inputs for all DMUs, and λ is a n-dimensional row vector of weights that forms convex combination of observed DMUs relative to which the subject DMU's efficiency is evaluated. The constraint in this problem simply describe the input requirement set as given by the observed data.

In basic models of DEA, we distinguish between input-oriented and output-oriented models. In other model, we combine both orientations in a single model, called Additive model. The Additive [2] model is presented as follows:

$$\begin{aligned} \max \ & \theta_t = es^- + es^+ \\ st: \ & \lambda Y - s^+ = y_t \\ & \lambda . X + s^- = x_t \\ & e\lambda = 1 \\ & \lambda \ge 0, \ s^-, s^+ \ge 0 \end{aligned} \tag{3}$$

Note that a DMU's efficiency is a relative measure. It compares a DMU's performance to the best practice performance implicit in the observed input-output combinations. If different input-output combinations were observed, a DMU's efficiency score would likely change. This idea is the bootstrap performed below.

3 The bootstrap

The essence of bootstrapping is to use computational power as a substitute for theoretical analysis. In this method, artificial, or pseudo-samples are drawn from the original data; the statistic is recalculated on the basis of each pseudo-sample; the resulting bootstrapped measures are then used to construct a sampling distribution for the statistic of interest. Note that in order for the bootstrap to work, the empirical distribution of the sample must be a good representation of the underlying population distribution that generated the sample in first place [5].

We use the efficiency scores calculated from the original data to form pseudo-samples of artificial data. Each artificial data set is similar to the original data set in that both follow the same distributions of inefficiency; this assures that the levels of performance within the bootstrapped results are within the realm of observed behavior.

The efficiency measures being considered in this article are input-based measures; the bootstrap is performed over the original efficiency scores. For this reason only the inputs are adjusted in the formation of the pseud-samples. The data in the pseudo-samples thus consist of the original output level for all n DMUs, the original input data for the DMU whose efficiency is being calculated, and adjusted input data for the remaining n-1 DMUs. After forming a pseudo-sample, the efficiency of a DMU's original input vector is then assessed relative to the technology implicit in it. Recalculating a DMU's efficiency relative to a large number of pseudo-samples generates a sampling distribution for the efficiency score.

To perform our analysis, we modify a form of the bootstrap that is commonly used in the analysis of regression equations. In this case we re-sample, with replacement, n-1 times from a uniform distribution over the set of original efficiency scores, $M^*=\{\theta_1^*,...,\theta_n^*\}$, produced by solving equation (3) once for each observation in the original data set. A set of pseudo-efficiency score, $M^b=\{\theta_1^b,...,\theta_{n-1}^b\}$, $\theta_j^b \in M^*$, j=1,…,n-1 are then used to construct a new reference technology relative to which efficiency is recalculated. Note that only n-1 pseudo-efficiency scores are drawn; we hold the efficiency of the DMU being assessed constant at its original value. A large number of pseudo-samples, say B, are formed, efficiency is calculated relative to each resulting pseudo-reference technology, and the empirical distribution for the efficiency measure is constructed from the resulting B efficiency scores. Note that a total of $B\times n$ pseudo-reference technologies and bootstrapped efficiency scores are generated in this process (B pseudo-samples are generated for each of the n observed DMUs in the data set). Specifically, the bootstrap we perform proceeds in four steps:

1) Solve equation (3) once for each DMU to obtain the set of empirical technical efficiency scores, $M^*=\{\theta_1^*,...,\theta_n^*\}$, based on the observed input and output data, X and Y.

2) Adjust the observed matrix of inputs, X by the calculated efficiency scores, to get a matrix of efficient inputs, $X^*=D.X$, where D is a $n\times n$ diagonal matrix as its elements: $\theta_1^*,...,\theta_n^*$ (observed efficiency scores).

3) For each DMU $t=1,...,n$:

(i) Draw, with replacement $n-1$ efficiency scores from the set M^* to get a pseudo-sample of efficiency scores, $M_t(b)=\{\theta_1^b,...,\theta_{t-1}^b,\theta_t^*,\theta_{t+1}^b,...,\theta_n^b\}$, $\theta_j^b \in M^*$, $j=1,...,t-1,t+1,...,n$. Note that the t^{th} DMU's efficiency score is maintained at its original level.

(ii) Construct a new matrix of observed pseudo-inputs as follows:

$$X_t(b)=[D_t(b)]^{-1}.X^* \tag{4}$$

where $D_t(b)$ is a $n\times n$ diagonal matrix containing the bootstrapped efficiency scores $\theta_1^b,...,\theta_{t-1}^b$, θ_t^*, $\theta_{t+1}^b,...,\theta_n^b$ as its diagonal elements. Note that some of the DMUs in the pseudo-sample will be efficient;

others will be inefficient. Further note that the t^{th} DMU's original input vector x_t will be contained in the t^{th} row of $X_t(b)$.

(iii) Calculate the technical efficiency of the t^{th} DMU relative to the pseudo-technology implicit in $X_t(b)$ and Y by solving the linear program:

$$\begin{aligned} \max \quad & \theta_t(b) = es^- + es^+ \\ st: \quad & \lambda Y - s^+ = y_t \\ & \lambda X_t(b) + s^- = x_t \\ & e\lambda = 1 \\ & \lambda \geq 0,\ s^-, s^+ \geq 0 \end{aligned}$$

to get the bootstrapped efficiency score $\theta_t^*(b)$.

Repeat steps (i)-(iii), B times to get the set of bootstrapped efficiency scores $\{\theta_t^*(1),...,\theta_t^*(B)\}$ for the t^{th} DMU.

4) Put $\theta_t^*(b)=[\theta_t^*(1)+...+\theta_t^*(B)]/B$, $(t=1,...,n)$ for each DMU.

By increasing the number of steps (B), distinct means of efficiency scores is obtained and this makes possible the ranking of DMUs.

4 An illustration using high-schools

The data used in this study are based on the data collection from 74 high-schools in the north of Iran. The high-schools used four inputs to produce three outputs. The results of the additive model and other methods for this inputs and outputs are summarized in table 1. This results does not supply much information to decision makers as it is not possible to distinguish among the performances of many of the high-schools. The bootstrap helps to shed more light upon the performance levels of the observed DMU.

Of the 74 high-schools in the original sample, 36 were found to operate on the best practice frontier ($\theta^*=0$). In this paper all DMUs has been ranked using Bootstrap Method. The main drawback in existing methods for ranking efficient DMU is non-extreme efficient DMU in which including such a DMUs do not alter PPS (Production Possibility Set), and methods can not be used for ranking them. The method is powerful in the sense that the repetition of the procure has no limitation. These DMUs have been ranked using AP [1], CSW [7], NORM1 [8], MAJ [9] and our method. The result is shown in table 1 which seems quite satisfactory.

Table 1 Results obtained by AP, CSW, NORM1, MAJ and our method

	Results use bootstrap method in the Additive model				Ranking by other methods			
DMUs	Original scores (θ_t^*)	Mean	Median	Ranking	AP	CSW	NORM 1	MAJ
1	131.3415	295.7869	285.1855	48	59	59	55	61
2	11.5334	181.0444	171.4837	32	45	44	38	50
3	0	27.2438	0	11	41	60	21	35
4	61.4810	231.5400	216.0695	40	32	28	43	32
5	0	11.0808	0	8	5	23	3	4
6	208.0993	314.8967	303.0562	52	48	54	66	55
7	0	198.5641	198.511	37	43	39	29	37
8	51.2502	341.4607	332.7379	56	25	19	38	25
9	317.8716	437.8243	422.6616	69	67	61	65	65

DMUs	Results use bootstrap method in the Additive model				Ranking by other methods			
	Original scores (θ_t^*)	Mean	Median	Ranking	AP	CSW	NORM 1	MAJ
10	418.1922	575.9821	547.7911	72	64	50	70	68
11	129.1713	309.3655	295.4552	51	37	20	53	39
12	289.3216	426.7248	412.7409	66	52	42	69	60
13	546.4621	607.1318	593.9522	73	70	74	74	73
14	244.3494	425.9299	411.8093	65	73	72	71	72
15	261.9596	516.2404	507.4521	71	72	68	45	71
16	0	27.2438	0	11	18	**	26	18
17	230.9844	387.8819	371.6051	61	38	22	57	41
18	127.7186	384.1676	363.3315	60	51	35	43	53
19	127.5180	326.7460	313.4282	55	68	67	52	69
20	182.6145	227.2144	219.1959	39	36	40	55	40
21	142.5254	325.3711	307.2538	54	71	69	60	70
22	42.7581	186.5603	175.5914	34	33	36	44	33
23	0	6.5172	0	6	13	11	23	13
24	0	32.6251	0	14	8	10	11	9
25	0	342.5704	330.4306	57	47	27	36	45
26	0	149.2227	0	30	17	32	17	17
27	0	0.3227	0	3	3	21	1	2
28	130.0709	192.0315	173.9406	36	28	18	37	28
29	226.0627	430.8881	417.3520	67	53	46	48	62
30	0	0.5198	0	5	2	**	8	3
31	0	407.1029	412.9359	64	74	73	32	74
32	0	264.8964	266.8159	46	22	24	33	22
33	0	59.3172	0	17	15	56	18	15
34	0	185.9570	171.8360	33	26	7	35	26
35	299.4050	380.3721	375.9367	59	65	63	68	66
36	202.7305	275.4625	267.9147	47	66	65	62	54
37	0	19.7869	0	9	7	**	10	6
38	0	188.1389	178.2205	35	29	14	28	30
39	0	139.4116	132.5936	26	24	9	27	24
40	67.8972	151.6794	141.3201	31	31	47	46	29
41	0	96.3132	89.7632	19	34	26	15	34
42	0	136.6381	138.248	25	57	57	16	48
43	66.6398	206.2988	194.0092	38	50	71	49	47
44	72.7699	147.7936	141.4345	29	27	12	40	27
45	0	108.0977	84.3604	22	14	**	20	12
46	0	10.4951	0	7	9	31	19	8
47	0	139.5439	134.1846	27	35	43	34	38
48	124.8167	236.6065	228.6723	41	61	64	50	52
49	0	45.0839	0	16	10	49	9	14
50	234.1879	305.2661	297.0225	50	60	55	72	58
51	0	34.8665	0	15	19	29	14	20
52	120.8030	259.5217	245.1513	45	44	34	47	44
53	0	22.1295	0	10	21	62	24	21
54	373.1328	470.6007	442.1664	70	58	53	63	63
55	270.0360	396.0343	378.0329	63	62	66	64	59
56	0	97.8545	100.1324	20	20	25	31	19
57	0	84.9918	64.6058	18	12	17	6	10
58	260.4233	392.9342	379.7449	62	46	30	67	49
59	190.6638	301.0276	295.0580	49	54	38	54	57
60	0	0.4227	0	4	6	15	4	7
61	275.9170	361.1228	357.3825	58	55	45	59	56
62	0	0.1221	0	1	1	8	2	1
63	0	141.3358	128.4637	28	40	52	25	42
64	347.4769	434.5885	406.3480	68	39	33	51	43
65	131.1285	247.5961	232.1854	43	30	13	41	31
66	0	101.6720	0	21	16	48	22	16
67	0	31.5332	0	13	11	**	7	11
68	175.2277	318.5097	298.9120	53	56	51	58	51
69	0	121.0747	95.9211	23	69	70	13	64
70	0	127.5704	123.3151	24	49	41	12	46

	Results use bootstrap method in the Additive model				Ranking by other methods			
DMUs	Original scores (θ_t^*)	Mean	Median	Ranking	AP	CSW	NORM 1	MAJ
71	0	0.221	0	2	4	**	5	5
72	594.6988	730.6583	704.4613	74	63	58	73	67
73	174.1718	237.4257	230.9656	42	42	37	61	36
74	0	251.9113	244.0799	44	23	16	30	23

5 Conclusion

This study proposed a procedure based on the Bootstrapping method to rank the all DMUs in non-radial models. The result is shown is Table 1. It can be seen that the difference between the results obtained by AP, CSW, NORM1, MAJ and our method is not significant.

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