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Fuzzy Efficiency Measures in DEA: A New Approach based on Fuzzy DEA Approach with Double Frontiers

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Abstract Data envelopment analysis (DEA) is a method to measure relative efficiency of a set of decision-making units (DMUs) which uses multiple inputs and produces multiple outputs. In the conventional DEA, crisp inputs and outputs are fundamentally necessary. But the observed values of inputs and outputs in real-world problems are sometimes imprecise. Thus, performance measurement often needs to be done under uncertainty conditions. This paper uses the DEA with double frontiers approach for selecting the best DMU in a fuzzy environment. In this approach, in addition to the optimistic fuzzy efficiency of each DMU, pessimistic fuzzy efficiency is considered. In contrast to the models of existing fuzzy DEA approaches, the proposed approach can accurately and easily identify the best DMU. The approach will be used to evaluate the performance of eight production units to demonstrate its simplicity and usefulness in selecting the best DMU.

Keywords: Data Envelopment Analysis, Fuzzy Input Data and Fuzzy Output Data, Optimistic and Pessimistic Fuzzy Efficiencies, Overall Performance.

1 Introduction

Data envelopment analysis (DEA) is a mathematical programming technique to measure the relative efficiency of decision-making units (DMUs) based on multiple inputs and multiple outputs [1,2]. The advantage of DEA is that it does not need any assumptions about the shape of frontier level and makes no assumptions about the internal operations of a DMU. It has successfully been used in many fields. As a result, a significant amount of research papers has been published in the DEA press [3-11].

The conventional DEA methods also need accurate measurement of inputs and outputs. But the observed values of input and output data in real-world problems are sometimes imprecise [12-17]. Some scholars proposed several fuzzy approaches to work with this inaccuracy in the DEA [18,19]. Since the initial study of Sengupta [20, 21], there has always been interest in fuzzy DEA papers and has been developing. The tolerance approach was one of the first fuzzy DEA models developed by Sengupta [20] and later improved by Kahraman and Tolga [22]. The α-level approach is the most common fuzzy DEA model. In this

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approach, the basic idea is that the fuzzy DEA model is converted into a pair of parameter programs to determine the lower and upper bounds of α-level of membership functions of efficiency scores. Kao and Liu [17] followed the basic idea of converting a fuzzy DEA model to a family of crisp conventional DEA models and developed a solution to measure the efficiency of DMUs with fuzzy observations in the BCC model. To deal with missing values, Kao and Liu [23] used a fuzzy DEA approach and calculated the efficiency scores of a set of DMUs with the α–level method proposed by Kao and Liu [17]. Karsak [24] proposed an extension of the model of Cook et al. [25] to assess the inputs and outputs of crisp, sequential and fuzzy data in flexible production systems by setting the optimistic and pessimistic α–level for the membership functions of efficiency scores. Guo et al. [26] for the first time developed the fuzzy DEA models based on the possibility and necessity measures, and then Lertworasirikul [27] and Lertworasirikul et al. [28],Lertworasirikul et al. [29] presented two approaches to solve the ranking problem in the fuzzy DEA models with possibility approach and necessity approach. Lertworasirikul et al. [30, 31] proposed a possibility approach to solve the fuzzy CCR model where fuzzy constraints are considered as fuzzy events. Garcia et al. [32] introduced a fuzzy DEA approach to ranking the identified failure modes by the incidence, severity and exposure indicators. Their method allows experts to use linguistic variables for assigning more important values to the considered indicators. Wen and Li [33] proposed a hybrid algorithm by combining fuzzy simulation and genetic algorithm to solve a fuzzy DEA model based on the credibility measure. Hougaard [34] extended the technical efficiency scores used in DEA to fuzzy intervals and showed that fuzzy scores allow decision-makers to use technical efficiency scores in combination with other sources of information on performance such as experts and key figures. Sheth and Triantis [35] introduced a fuzzy goal DEA framework to measure and assess the efficiency and effectiveness of goals in a fuzzy environment. They defined a membership function for each fuzzy constraint along with efficiency and effectiveness goals and showed the degree of success of the constraint. Wang et al. [36] proposed a pair of interval DEA models to work with imprecise data such as internal data, ordinal preference information, fuzzy data, and their combination. In their approach, efficiency scores were obtained as interval numbers. Luban [37] proposed a method inspired by the work of Sheth and Triantis [35] and used the fuzzy aspect of DEA models to select the membership function, the bounds of inputs and outputs, global goals, and bound of global goals. Wang et al. [18] presented two fuzzy DEA models with fuzzy inputs and outputs by fuzzy arithmetic. They converted each of the proposed fuzzy CCR models into three linear programming (LP) models to calculate the efficiency of DMUs as fuzzy numbers. Qin and Liu [38] developed a bunch of random fuzzy DEA models with fuzzy inputs and outputs where randomness and fuzziness exist at the same time in the evaluation system, and fuzzy random data was determined with the possibility and probability distribution.

In this paper, we introduce a new method called fuzzy DEA approach with double frontiers to select the best DMU [39,40]. The approach considers two efficiencies for decision-making: one is measured against the efficiency frontier and is called the best fuzzy efficiency or optimistic fuzzy efficiency, and the other against inefficiency frontier (or input frontier) and is called the worst fuzzy efficiency or pessimistic fuzzy efficiency. The traditional fuzzy DEA only measures the best fuzzy efficiencies of a group of DMUs while ignoring pessimistic fuzzy efficiencies. Therefore, it cannot provide an overall assessment of DMUs. With simultaneous consideration of optimistic and pessimistic fuzzy efficiencies, one can identify the best DMU [41,42]. This will be shown in Section 3 with a numerical example.

The paper is organized as follows. Section 2 introduces the fuzzy DEA models for measuring optimistic and pessimistic fuzzy efficiencies of DMUs, and then it suggests overall performance measures. The numerical example is discussed in Section 3, and Section 4 describes the final considerations.

2 Fuzzy DEA models to measure optimistic and pessimistic fuzzy efficiencies

A fuzzy number is a fuzzy convex set characterized by a specific range of real numbers, each of which has a degree of membership between zero and one. The most common fuzzy numbers are triangular and trapezoidal fuzzy numbers whose membership functions are defined as follows:

$$\mu_{\tilde{A}_1}(x)=\begin{cases}(x-a)/(b-a), & a\le x\le b,\\ (d-x)/(d-b), & b\le x\le d,\\ 0, & \text{otherwise},\end{cases} \tag{1}$$

$$\mu_{\tilde{A}_2}(x)=\begin{cases}(x-a)/(b-a), & a\le x\le b,\\ 1, & b\le x\le c,\\ (d-x)/(d-c), & c\le x\le d,\\ 0, & \text{otherwise}.\end{cases} \tag{2}$$

For summary, the triangular or trapezoidal fuzzy numbers are often shown as (a,b,d) and (a,b,c,d).

Assume $\tilde{A}=(a_L,a_M,a_U)$ and $\tilde{B}=(b_L,b_M,b_U)$ are two positive triangular fuzzy numbers. Basic fuzzy arithmetic operations on these fuzzy numbers are defined as follows:

Addition: $\tilde{A}+\tilde{B}=(a_L+b_L,a_M+b_M,a_U+b_U)$;

Subtraction: $\tilde{A}-\tilde{B}=(a_L-b_U,a_M-b_M,a_U-b_L)$;

Multiplication: $\tilde{A}\times\tilde{B}\approx(a_Lb_L,a_Mb_M,a_Ub_U)$;

Division: $\tilde{A}/\tilde{B}\approx\left(\frac{a_L}{b_U},\frac{a_M}{b_M},\frac{a_U}{b_L}\right)$.

2.1 Fuzzy DEA models to measure optimistic fuzzy efficiencies of DMUs

Assume that there are n DMUs for evaluation, and each DMU consists of m inputs and s outputs. We define x_{ij} ($i=1,\ldots,m$) and y_{rj} ($r=1,\ldots,s$) as the input and output values of DMU_j ($j=1,\ldots,n$). Without loss of generality, it is assumed that all input and output data x_{ij} and y_{rj} are characterized by triangular fuzzy numbers $\tilde{x}_{ij}=(x_{ij}^L,x_{ij}^M,x_{ij}^U)$ and $\tilde{y}_{rj}=(y_{rj}^L,y_{rj}^M,y_{rj}^U)$ where $x_{ij}^L>0$ and $y_{rj}^L>0$ for $i=1,\ldots,m$, $r=1,\ldots,s$ and $j=1,\ldots,n$. Crisp input and output data can be seen as a special case of fuzzy input and output data $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ with $x_{ij}^L=x_{ij}^M=x_{ij}^U$ and $y_{rj}^L=y_{rj}^M=y_{rj}^U$. The efficiency of DMU_j is defined as follows:

$$\widetilde{\theta}_j = \frac{\sum_{r=1}^{s} u_r \widetilde{y}_{rj}}{\sum_{i=1}^{m} v_i \widetilde{x}_{ij}}, \tag{3}$$

Where is a fuzzy number called fuzzy efficiency, and u_r $(r = 1,\ldots,s)$ and v_i $(i = 1,\ldots,m)$ respectively are weights assigned to outputs and inputs.

To work with such an uncertain situation, Wang et al. [18] presented these LP models to obtain fuzzy efficiency, which measures the optimistic fuzzy efficiency of DMUs:

$$\begin{aligned} \max \quad & \theta_o^U = \sum_{r=1}^{s} u_r y_{ro}^U \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io}^L = 1, \\ & \sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i x_{ij}^L \le 0, \quad j = 1,\ldots,n, \\ & u_r, v_i \ge 0, \quad i = 1,\ldots,m; \quad r = 1,\ldots,s \end{aligned} \tag{4}$$

$$\begin{aligned} \max \quad & \theta_o^M = \sum_{r=1}^{s} u_r y_{ro}^M \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io}^M = 1, \\ & \sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i x_{ij}^L \le 0, \quad j = 1,\ldots,n, \\ & u_r, v_i \ge 0, \quad i = 1,\ldots,m; \quad r = 1,\ldots,s \end{aligned} \tag{5}$$

$$\begin{aligned} \max \quad & \theta_o^L = \sum_{r=1}^{s} u_r y_{ro}^L \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io}^U = 1, \\ & \sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i x_{ij}^L \le 0, \quad j = 1,\ldots,n, \\ & u_r, v_i \ge 0, \quad i = 1,\ldots,m; \quad r = 1,\ldots,s \end{aligned} \tag{6}$$

where DMU_o represents the DMU under evaluation. The optimal values of the objective function of three LP models (4)-(6) constitute the best fuzzy efficiency of DMU_o. That is, $\widetilde{\theta}_o^* \approx (\theta_o^{L*}, \theta_o^{M*}, \theta_o^{U*})$, which can be almost seen as a triangular fuzzy number. If there is a set of positive weights u_r^* $(r = 1,\ldots,s)$ and v_i^* $(i = 1,\ldots,m)$ that makes $\theta_o^{U*} = 1$, then DMU_o is called fuzzy DEA efficient or optimistic efficient; otherwise, fuzzy DEA non-efficient or optimistic non-efficient. All optimistic efficient DMUs together form an efficiency frontier.

2.2 Fuzzy DEA models to measure pessimistic fuzzy efficiencies of DMUs

The framework with an input nature, based on a set of input requirements and its inefficiency frontier, seeks to maintain output in the current limit, while maximizing input values. It emphasizes the fact that the output level remains unchanged, and input values are proportionally increased until the inefficient production frontier is reached. The DEA estimator for the inefficient production possibility set is called pessimistic efficiency or the worst relative efficiency. For a particular DMU, for example DMU_o, fuzzy efficiencies can be calculated from the following pessimistic fuzzy DEA models [43]:

$$\begin{aligned}
\min \quad & \varphi_o^L = \sum\nolimits_{r=1}^{s} u_r y_{ro}^L \\
\text{s.t.} \quad & \sum\nolimits_{i=1}^{m} v_i x_{io}^U = 1, \\
& \sum\nolimits_{r=1}^{s} u_r y_{rj}^L - \sum\nolimits_{i=1}^{m} v_i x_{ij}^U \geq 0, \quad j = 1,\ldots,n, \\
& u_r, v_i \geq 0, \quad i = 1,\ldots,m; \quad r = 1,\ldots,s
\end{aligned} \tag{7}$$

$$\begin{aligned}
\min \quad & \varphi_o^M = \sum\nolimits_{r=1}^{s} u_r y_{ro}^M \\
\text{s.t.} \quad & \sum\nolimits_{i=1}^{m} v_i x_{io}^M = 1, \\
& \sum\nolimits_{r=1}^{s} u_r y_{rj}^L - \sum\nolimits_{i=1}^{m} v_i x_{ij}^U \geq 0, \quad j = 1,\ldots,n, \\
& u_r, v_i \geq 0, \quad i = 1,\ldots,m; \quad r = 1,\ldots,s
\end{aligned} \tag{8}$$

$$\begin{aligned}
\min \quad & \varphi_o^U = \sum\nolimits_{r=1}^{s} u_r y_{ro}^U \\
\text{s.t.} \quad & \sum\nolimits_{i=1}^{m} v_i x_{io}^L = 1, \\
& \sum\nolimits_{r=1}^{s} u_r y_{rj}^L - \sum\nolimits_{i=1}^{m} v_i x_{ij}^U \geq 0, \quad j = 1,\ldots,n, \\
& u_r, v_i \geq 0, \quad i = 1,\ldots,m; \quad r = 1,\ldots,s
\end{aligned} \tag{9}$$

When there is a set of positive weights u_r^* $(r = 1,\ldots,s)$ and v_i^* $(i = 1,\ldots,m)$ that makes $\varphi_o^{L*} = 1$, we say that DMU_o is fuzzy DEA inefficient or pessimistic inefficient. Otherwise, we say that it is fuzzy DEA non-inefficient or pessimistic inefficient. All pessimistic inefficient DMUs form an inefficiency frontier. The optimal values of the objective function of three LP models (7)-(9) form the worst fuzzy efficiency of DMU_o. That is, $\tilde{\varphi}_o^* \approx (\varphi_o^{L*}, \varphi_o^{M*}, \varphi_o^{U*})$, which can be seen almost as a triangular fuzzy number.

Optimistic and pessimistic efficiencies measure two performance bases of each DMU. Any evaluation method that considers only either one gets biased [44-47]. To determine the overall performance of each DMU, both efficiencies should be considered at the same time.

2.3 Overall performance measures

Optimistic and pessimistic fuzzy efficiencies are measured from different perspectives, resulting in two different scoring for DMUs. Therefore, an overall performance measure is needed to obtain the overall score of DMUs. Here, we use the overall performance measure proposed by Wang and Chin [48] for scoring DMUs with crisp data as follows:

$$\phi_j = \frac{\theta_j^*}{\sqrt{\sum_{i=1}^{n} \theta_i^{*2}}} + \frac{\varphi_j^*}{\sqrt{\sum_{i=1}^{n} \varphi_i^{*2}}}, \quad j = 1,\ldots,n \tag{10}$$

where θ_j^* and φ_j^* respectively are optimistic and pessimistic efficiencies of DMU_j. It is clear that the overall performance measure defined in (10) considers the magnitude of two efficiencies.

Assume that $\tilde{\theta}_j^* \approx (\theta_j^{L*}, \theta_j^{M*}, \theta_j^{U*})$ and $\tilde{\varphi}_j^* \approx (\varphi_j^{L*}, \varphi_j^{M*}, \varphi_j^{U*})$ respectively are optimistic and pessimistic fuzzy efficiencies of DMU_j. According to the operating rules on fuzzy data, we have:

$$\begin{aligned}
\phi_j &\approx \frac{(\theta_j^{L*},\theta_j^{M*},\theta_j^{U*})}{\sqrt{\sum_{i=1}^{n}(\theta_i^{L*},\theta_i^{M*},\theta_i^{U*})^2}}+\frac{(\varphi_j^{L*},\varphi_j^{M*},\varphi_j^{U*})}{\sqrt{\sum_{i=1}^{n}(\varphi_i^{L*},\varphi_i^{M*},\varphi_i^{U*})^2}} \\
&\approx \frac{(\theta_j^{L*},\theta_j^{M*},\theta_j^{U*})}{\sqrt{(\sum_{i=1}^{n}\theta_i^{L*2},\sum_{i=1}^{n}\theta_i^{M*2},\sum_{i=1}^{n}\theta_i^{U*2})}}+\frac{(\varphi_j^{L*},\varphi_j^{M*},\varphi_j^{U*})}{\sqrt{(\sum_{i=1}^{n}\varphi_i^{L*2},\sum_{i=1}^{n}\varphi_i^{M*2},\sum_{i=1}^{n}\varphi_i^{U*2})}} \\
&\approx \left(\frac{\theta_j^{L*}}{\sqrt{\sum_{i=1}^{n}\theta_i^{U*2}}},\frac{\theta_j^{M*}}{\sqrt{\sum_{i=1}^{n}\theta_i^{M*2}}},\frac{\theta_j^{U*}}{\sqrt{\sum_{i=1}^{n}\theta_i^{L*2}}}\right)+\left(\frac{\varphi_j^{L*}}{\sqrt{\sum_{i=1}^{n}\varphi_i^{U*2}}},\frac{\varphi_j^{M*}}{\sqrt{\sum_{i=1}^{n}\varphi_i^{M*2}}},\frac{\varphi_j^{U*}}{\sqrt{\sum_{i=1}^{n}\varphi_i^{L*2}}}\right) \\
&\approx \left(\frac{\theta_j^{L*}}{\sqrt{\sum_{i=1}^{n}\theta_i^{U*2}}}+\frac{\varphi_j^{L*}}{\sqrt{\sum_{i=1}^{n}\varphi_i^{U*2}}},\frac{\theta_j^{M*}}{\sqrt{\sum_{i=1}^{n}\theta_i^{M*2}}}+\frac{\varphi_j^{M*}}{\sqrt{\sum_{i=1}^{n}\varphi_i^{M*2}}},\frac{\theta_j^{U*}}{\sqrt{\sum_{i=1}^{n}\theta_i^{L*2}}}+\frac{\varphi_j^{U*}}{\sqrt{\sum_{i=1}^{n}\varphi_i^{L*2}}}\right), \quad j=1,\ldots,n.
\end{aligned} \tag{11}$$

It is obvious that ϕ_j ($j=1,\ldots,n$) should be a fuzzy number shown with $\phi_j \approx (\phi_j^L,\phi_j^M,\phi_j^U)$ ($j=1,\ldots,n$). In this case, we have:

$$\begin{aligned}
\phi_j^L &\approx \frac{\theta_j^{L*}}{\sqrt{\sum_{i=1}^{n}\theta_i^{U*2}}}+\frac{\varphi_j^{L*}}{\sqrt{\sum_{i=1}^{n}\varphi_i^{U*2}}}, \quad j=1,\ldots,n, \\
\phi_j^M &\approx \frac{\theta_j^{M*}}{\sqrt{\sum_{i=1}^{n}\theta_i^{M*2}}}+\frac{\varphi_j^{M*}}{\sqrt{\sum_{i=1}^{n}\varphi_i^{M*2}}}, \quad j=1,\ldots,n, \\
\phi_j^U &\approx \frac{\theta_j^{U*}}{\sqrt{\sum_{i=1}^{n}\theta_i^{L*2}}}+\frac{\varphi_j^{U*}}{\sqrt{\sum_{i=1}^{n}\varphi_i^{L*2}}}, \quad j=1,\ldots,n.
\end{aligned} \tag{11}$$

For convenience, an approach that determines the overall performance of each DMU against both optimistic and pessimistic fuzzy efficiencies is called fuzzy DEA approach with double frontiers [41,49,50]. The efficiency frontier of a set of optimistic efficient DMUs determines that they have relatively good performance, while the inefficiency frontier of a set of pessimistic inefficient DMUs determines that they have relatively poor performance. The best DMU can usually be selected among optimistic efficient DMUs.

Since the final efficiency score of each DMU is determined by a fuzzy number, a simple but applied rankings approach is needed to compare and ranking the efficiencies of DMUs. Several approaches have been developed previously for rating fuzzy numbers, but all have disadvantages. In this paper, we use the “degree of preference approach” (developed by Wang et al. [18]) to compare and ranking the fuzzy efficiency of DMUs.

3 A practical example

Consider a performance evaluation problem in China where eight manufacturing enterprises (DMUs) should be evaluated in terms of two inputs and two outputs. The dataset for this analysis is taken from the paper of Wang et al. [18]. The eight manufacturing enterprises the same product but with different qualities. *Gross output value* and *product quality* are considered as output. *Manufacturing cost* and *number of employees* are considered as input. Data on gross output value and manufacturing cost is unknown due to the absence at the time of measurement, and thus, they are evaluated as fuzzy numbers. Product quality is measured by customers using fuzzy linguistic terms such as *excellent*, *very good*, *good*, *average*, *poor*, or *very poor*. The results of customer evaluation are weighted and averaged. Table 1 shows input and output data for eight manufacturing enterprises.

Table 1 input and output data for eight manufacturing enterprises

Enterprises (DMUs)	Inputs		Outputs	
	Manufacturing cost	Number of employees	Gross output value	Product quality
A	(2120, 2170, 2210)	1870	(14500, 14790, 14860)	(3.1, 4.1, 4.9)
B	(1420, 1460, 1500)	1340	(12470, 12720, 12790)	(1.2, 2.1, 3.0)
C	(2510, 2570, 2610)	2360	(17900, 18260, 18400)	(3.3, 4.3, 5.0)
D	(2300, 2350, 2400)	2020	(14970, 15270, 15400)	(2.7, 3.7, 4.6)
E	(1480, 1520, 1560)	1550	(13980, 14260, 14330)	(1.0, 1.8, 2.7)
F	(1990, 2030, 2100)	1760	(14030, 14310, 14400)	(1.6, 2.6, 3.6)
G	(2200, 2260, 2300)	1980	(16540, 16870, 17000)	(2.4, 3.4, 4.4)
H	(2400, 2460, 2520)	2250	(17600, 17960, 18100)	(2.6, 3.6, 4.6)

By solving the fuzzy DEA models (4)-(6) respectively for each of the manufacturing enterprises, their optimistic fuzzy efficiency scores are obtained, as shown in the second column of Table 2. According to Table 2, three manufacturing enterprises (A, B and E), based on the fuzzy DEA model (4), are fuzzy DEA efficient or optimistic efficient and they together defined an efficient production frontier. The remaining five companies with fuzzy efficiency scores less than 1 are considered optimistic efficient. To provide full performance ranking for these eight manufacturing enterprises from the optimistic viewpoint, Table 3 shows the degree of preference matrix calculated for optimistic fuzzy efficiencies and their rankings.

Table 2 Optimistic and pessimistic fuzzy efficiencies for eight manufacturing enterprises

DMUs	Models (4)–(6)			Models (7)–(9)		
	Lower bound (θ_j^{L*})	Middle value (θ_j^{M*})	Upper bound (θ_j^{U*})	Lower bound (φ_j^{L*})	Middle value (φ_j^{M*})	Upper bound (φ_j^{U*})
A	0.8124	0.9033	1.0000	1.0463	1.0672	1.0723
B	0.9750	0.9945	1.0000	1.1106	1.2602	1.2879
C	0.7946	0.8122	0.9045	1.0235	1.0440	1.0520
D	0.7764	0.8050	0.9070	1.0000	1.0200	1.0287
E	0.9603	0.9872	1.0000	1.0000	1.1701	1.2475
F	0.8352	0.8518	0.8852	1.0000	1.0971	1.1040
G	0.8752	0.8927	0.9457	1.0850	1.1497	1.1585
H	0.8195	0.8363	0.8864	1.0190	1.0771	1.0855

According to Table 3, the eight companies are ranked in terms of the best fuzzy efficiency as follows:

$$B \overset{70.09\%}{\succ} E \overset{94.39\%}{\succ} A \overset{52.16\%}{\succ} G \overset{98.37\%}{\succ} F \overset{66.38\%}{\succ} H \overset{62.54\%}{\succ} C \overset{56.08\%}{\succ} D$$

where $B \overset{70.09\%}{\succ} E$ means that the performance of B is better than E by 70.09%. It is clear that B has the best performance followed by E, A and G.

Table 3 Matrix of degree of preference for optimistic fuzzy efficiencies obtained based on models (4)-(6) and their rankings

DMUs	A	B	C	D	E	F	G	H	Rank
A	–	0.0253	0.8444	0.8542	0.0561	0.8205	**0.5216**	0.8474	3
B	0.9747	–	1.0000	1.0000	**0.7009**	1.0000	1.0000	1.0000	1
C	0.1556	0.0000	–	**0.5608**	0.0000	0.2758	0.0433	0.3746	7
D	0.1458	0.0000	0.4392	–	0.0000	0.2407	0.0421	0.3263	8
E	**0.9439**	0.2991	1.0000	1.0000	–	1.0000	1.0000	1.0000	2
F	0.1795	0.0000	0.7242	0.7593	0.0000	–	0.0163	**0.6638**	5
G	0.4784	0.0000	0.9567	0.9579	0.0000	**0.9837**	–	0.9865	4
H	0.1526	0.0000	**0.6254**	0.6737	0.0000	0.3362	0.0135	–	6

By solving the fuzzy DEA models (7)-(9), the pessimistic fuzzy efficiency scores of the manufacturing enterprises are obtained, as shown in the third column of Table 2. From the perspective of pessimistic fuzzy efficiency, three companies (D, E and F) are fuzzy DEA inefficient or pessimistic inefficient based on the fuzzy DEA model (7). The remaining five manufacturing enterprises, with fuzzy efficiency scores more than 1, are considered pessimistic inefficient. Table 4 shows the degree of preference matrix calculated for pessimistic fuzzy efficiencies and their rankings from the pessimistic perspective.

Table 4 Matrix of degree of preference for pessimistic fuzzy efficiencies obtained based on models (7)-(9) and their rankings

DMUs	A	B	C	D	E	F	G	H	Rank
A	–	0.0000	**0.9791**	1.0000	0.1090	0.3933	0.0000	0.4863	6
B	1.0000	–	1.0000	1.0000	**0.8055**	1.0000	0.9421	1.0000	1
C	0.0209	0.0000	–	**0.9835**	0.0551	0.1943	0.0000	0.1740	7
D	0.0000	0.0000	0.0165	–	0.0167	0.0587	0.0000	0.0149	8
E	0.8910	0.1945	0.9449	0.9833	–	0.8261	**0.5625**	0.8696	2
F	0.6067	0.0000	0.8057	0.9413	0.1739	–	0.0284	**0.5939**	4
G	1.0000	0.0579	1.0000	1.0000	0.4375	**0.9716**	–	1.0000	3
H	**0.5137**	0.0000	0.8260	0.9851	0.1304	0.4061	0.0000	–	5

According to Table 4, the eight manufacturing enterprises, in terms of pessimistic fuzzy efficiency, are ranked as follows:

$$B \overset{80.55\%}{\succ} E \overset{56.25\%}{\succ} G \overset{97.16\%}{\succ} F \overset{59.39\%}{\succ} H \overset{51.37\%}{\succ} A \overset{97.91\%}{\succ} C \overset{98.35\%}{\succ} D$$

It is clear that B has the best performance followed by E, G and F.

The above measurements are based on different views and thus they may have different results. For example, the manufacturing enterprise A when evaluated from the optimistic perspective is considered optimistic efficient and ranks third. However, when evaluated from a pessimistic view, it is pessimistic inefficient and ranks sixth, that is, its performance is lower than all other manufacturing enterprises (excluding C and D). These two evaluation results certainly conflict with each other. Any evaluation result that considers only one of these two perspectives would undoubtedly be one-sided, unrealistic and non-persuasive [51-55].

Overall performance measures (12) consider not only optimistic fuzzy efficiencies, but also pessimistic fuzzy efficiencies and therefore are more comprehensive, and can be used as overall fuzzy efficiency measures for any manufacturing enterprise. According to overall performance measures (12), the eight manufacturing enterprises efficiency scores are shown in Table 5.

Table 5 Evaluation of eight manufacturing enterprises using fuzzy DEA approach with double frontiers

DMUs	Overall performance measures (12)		
	Lower bound (ϕ_j^L)	Middle value (ϕ_j^M)	Upper bound (ϕ_j^U)
A	0.6313	0.6986	0.7775
B	0.7124	0.7962	0.8510
C	0.6175	0.6550	0.7312
D	0.6033	0.6445	0.7243
E	0.6723	0.7647	0.8372
F	0.6254	0.6876	0.7410
G	0.6669	0.7206	0.7845
H	0.6254	0.6751	0.7352

The last column of Table 6 shows the ranking of the eight manufacturing enterprises based on overall performance measures. It is clear that the manufacturing enterprise B has the best overall performance. Although the manufacturing enterprise A is optimistic efficient and its overall performance is not better than G. While the latter is neither optimistic efficient nor pessimistic inefficient, its overall performance is better. And finally, the eight manufacturing enterprises are ranked based on overall performance measures are as follows:

$$B \overset{67.16\%}{\succ} E \overset{71.49\%}{\succ} G \overset{65.05\%}{\succ} A \overset{61.91\%}{\succ} F \overset{56.27\%}{\succ} H \overset{60.23\%}{\succ} C \overset{58.56\%}{\succ} D$$

Table 6 Matrix of degree of preference obtained for overall performance measures based on equations (12) and their rankings

DMUs	A	B	C	D	E	F	G	H	Rank
A	–	0.0914	0.7324	0.7799	0.2075	**0.6191**	0.3495	0.6690	4
B	0.9086	–	0.9912	0.9967	**0.6716**	0.9765	0.8625	0.9854	1
C	0.2676	0.0088	–	**0.5856**	0.0738	0.3527	0.1375	0.3977	7
D	0.2201	0.0033	0.4144	–	0.0549	0.2911	0.1033	0.3272	8
E	0.7925	0.3284	0.9262	0.9451	–	0.8847	**0.7149**	0.9056	2
F	0.3809	0.0235	0.6473	0.7089	0.1153	–	0.2198	0.5627	5
G	**0.6505**	0.1375	0.8625	0.8967	0.2851	0.7802	–	**0.8198**	3
H	0.3310	0.0146	**0.6023**	0.6728	0.0944	0.4373	0.1802	–	6

4 Conclusion

The fuzzy set theory is widely used for modeling uncertainty in DEA. Applications of fuzzy set theory in DEA are usually divided into four categories: (1) the tolerance approach, (2) the α-level based approach, (3) the fuzzy ranking approach, and (4) the possibility approach [9].

In this paper, we proposed the fuzzy DEA approach with double frontiers to select the best DMU. The proposed approach considers not only the optimistic fuzzy efficiencies of DMUs, but also their pessimistic fuzzy efficiencies. We also proposed overall performance measures to integrate the two different fuzzy efficiencies of each DMU. The proposed overall performance measures consider the magnitude of two different fuzzy efficiencies. To show the simplicity and usefulness of the proposed approach, it was used for evaluating the performance of manufacturing enterprises in China. It was shown that the proposed approach has a significant advantage over current methods for evaluating DMUs. It is expected that the proposed fuzzy DEA approach will have an important role in selecting the best DMU, and more applications in the future.

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