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A model for finding the most efficient DMU in DEA

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Abstract Finding the most efficient DMU in DEA is an important issue for decision makers. In this paper, we consider some of existing models which try to find the most efficient DMU and we present some of their drawbacks. Then, we propose a model which provides more discrimination between DMUs. This model tries to find a hyperplane which pass through a DMU with large distance from the others, and a single unit then turns out to be the most efficient one. Finally, we check the model by a numerical example.

Keyword: Data Envelopment Analysis, The Most Efficient DMU, Hyperplane.

1 Introduction

Data envelopment analysis (DEA) is a mathematical method to determine the productive efficiency of decision making units (DMUs) which is introduced by Charnes, Cooper, and Rhodes (CCR) [1]. The issue is followed by Banker, Charnes and Cooper (BCC) [2] to the more general model. When DEA is implemented to select only one DMU among existing DMUS, as efficient one, more discrimination may be necessary. In [3], Amin and Toloo studied a linear model to determine the most efficient DMU. Amin [4] shows that the model may have alternative optimal solution and introduced a nonlinear model in order to solve this problem. Foroughi [5] then showed that the model also maybe infeasible and introduced a mixed integer linear model to determine a single efficient DMU.

To extend the approach of Amin and Toloo [3] for BCC model, Toloo and Nalchiger [6] introduced a model to select most BCC-efficient DMU, however Foroughi [7] shows that the proposed model of Toloo and Nalchigar [6] for BCC models may be infeasible and fails to provide any relevant result, so introduced a model to solve the infeasibility problem for selecting the most BCC-efficient DMU. In this paper, it will be shown that the proposed model of Foroughi [7] may have alternative optional solution and then it could not select a single DMU, as efficient one, among the other DMUs.

In order to overcome this problem of the proposed model in [7], we introduced a model to solve the infeasibility problem and find the most efficient unit by using hyperplanes.

The structure of the paper is organized as follows. In Section 2, we have briefly reviewed some existing models. In Section 3, a new model for finding the most efficient DMU is

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proposed. A numerical example is presented in Section 4. The final section devoted to the conclusions.

2 Data envelopment analysis and the most efficient unit

DEA is implemented to evaluate the relative efficiency of DMUs in order to find the most efficient ones. The so-called CCR model, proposed by Chernes et al. [1], is as follows:

$$\begin{aligned}
&\text{Max} \sum_{r=1}^{s} u_r y_{rp} \\
&\text{s.t.} \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le o\ , \ j=1,\dots,n \\
&\sum_{i=1}^{m} v_i x_{ip} = 1 \\
&u_r \ge o,\ v_i \ge o \quad r=1,\dots,s,\ i=1,\dots,m
\end{aligned} \tag{1}$$

where ($j=1,2,\dots,n$) for n DMUs which consume m inputs ($x_i : i=1,2,\dots,m$) to produce s outputs ($y_r : i=1,2,\dots,m$). The most well-known extension of CCR model is the BCC model which is introduced by Banker et al. [2] through solving the following linear program:

$$\begin{aligned}
&\text{Max} \sum_{r=1}^{s} u_r y_{rj} + u_o \\
&\text{s.t.} \\
&\sum_{i=1}^{m} v_i x_{io} = 1 \\
&\sum_{r=1}^{s} u_r y_{rj} - u_o - \sum_{i=1}^{m} v_i x_{ij} \le o\ , \quad j=1,\dots,n \\
&u_o \quad \text{free}, \\
&u_r \ge o, \quad r=1,\dots,s, \\
&v_i \ge o, \quad i=1,\dots,m.
\end{aligned} \tag{2}$$

Where u_r and v_i are the weights of x_{ij} and y_{rj} which are the inputs and outputs of DMU_j, respectively. x_{io} and y_{ro} are the associated inputs and outputs of DMU_o.

Moreover, Ertay et al. [8], based on the DEA/AHP methodology, developed a robust layout framework to evaluate the facility layout design in order to find single most efficient DMU. The model is given by:

$$\begin{aligned}
&\text{Min } M - kd_o \\
&\text{s.t.} \\
&\sum_{i=1}^{m} w_i x_{ij} \le 1, \qquad j=1,\dots,n \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} w_i x_{ij} + d_j = 0, \qquad j=1,\dots,n
\end{aligned} \tag{3}$$

$$M - d_j \geq 0, \qquad j = 1, \ldots, n$$
$$\mu_r, w_i d_j \geq \varepsilon > 0, \; r = 1, \ldots, s \;, i = 1, \ldots, m \;, j = 1, \ldots, n$$

where M is the maximum value of all deviation variables and $k \in (0,1)$ is a constant which should be determined by trial-and-error to obtain a single relatively efficient DMU.

Amin & Toloo [3] is shown that model (3) may obtain more than one DMU as efficient, and then they suggest an improved model to find the most efficient DMU. This new model does not need to fix the parameter k through adding the constraint $\sum_{j=1}^{n} d_j = n - 1$ as:

$$\begin{aligned}
&\text{Min } M \\
&s.t. \\
&M - d_j \geq 0, \quad j = 1, \ldots, n \\
&\sum_{i=1}^{m} w_i x_{ij} \leq 1, \quad j = 1, \ldots, n \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} w_i x_{ij} + d_j - \beta_j = 0, \quad j = 1, \ldots, n \\
&\sum_{j=1}^{n} d_j = n - 1 \\
&0 \leq \beta_j \leq 1, \quad d_j \epsilon \{0,1\}, \quad j = 1, \ldots, n \\
&w_i \geq \varepsilon^* \quad i = 1, \ldots, m \\
&u_r \geq \varepsilon^* \quad r = 1, \ldots, s
\end{aligned} \tag{4}$$

which ε^* is the maximum non-Archimedean value and $d_j \epsilon \{0,1\}$ makes only one hyperplane be dominant. Although this model tries to find the most efficient unit, Amin [4] showed that it may obtain more than one efficient DMU and proposed a revised model as follows:

$$\begin{aligned}
&\text{Min } M \\
&s.t. \\
&M - d_j \geq 0, \quad j = 1, \ldots, n \\
&\sum_{i=1}^{m} w_i x_{ij} \leq 1, \quad j = 1, \ldots, n \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} w_i x_{ij} + d_j = 0, \quad j = 1, \ldots, n \\
&\sum_{j=1}^{n} \theta_j = n - 1, \frac{n!}{r!(n-r)!} \\
&\theta_j - d_j \beta_j = 0, \qquad j = 1, \ldots, n \\
&d_j \geq 0, \quad \beta_j \geq 1, \quad \theta_j \epsilon \{0,1\}, \quad j = 1, \ldots, n \\
&w_i \geq \varepsilon^* \quad i = 1, \ldots, m \\
&u_r \geq \varepsilon^* \quad r = 1, \ldots, s
\end{aligned} \tag{5}$$

Note that this model omits β_j values by adding new binary values of θ_j and changes the constraint $0 \le \beta_j \le 1$ to $\beta_j \ge 1, \quad j = 1, \dots, n$ in order to determine a single unit as the most efficient one.

Toloo and Nalchigar [6] extended the approach of Amin and Toloo [3] for the BCC models and try to find most BCC-efficient DMU. The model formulated as follows:

$$
\begin{aligned}
& M^* = \text{Min} \;\; M \\
& s.t. \\
& M - d_j \ge 0, \quad j = 1, \dots, n \\
& \sum_{i=1}^{m} w_i x_{ij} \le 1, \quad j = 1, \dots, n \\
& \sum_{r=1}^{s} u_r y_{rj} - u_0 - \sum_{i=1}^{m} w_i x_{ij} + d_j - \beta_j = 0, \quad j = 1, \dots, n \qquad (6) \\
& \sum_{j=1}^{n} d_j = n - 1 \\
& 0 \le \beta_j \le 1, \quad d_j \epsilon \{0,1\}, \; j = 1, \dots, n \\
& M, \; u_o \;\; \textit{free} \\
& w_i \ge \varepsilon^* \qquad i = 1, \dots, m \\
& u_r \ge \varepsilon^* \qquad r = 1, \dots, s
\end{aligned}
$$

By using this model, DMU_j introduced as the most efficient unit if and only if $d_j = 0$. Also, they showed that the model is feasible and could find a single unit as the most efficient one.

Foroughi [7] claimed that the model (6) is infeasible and has an alternative optimal solution which could not discriminate one unit as the most efficient one. In order to overcome this problem, a new model is proposed as:

$$
\begin{aligned}
& \varepsilon'^* = \text{Max} \;\; \varepsilon^* \\
& s.t. \\
& \sum_{i=1}^{m} w_i x_{ij} \le 1, \quad j = 1, \dots, n, \\
& \sum_{r=1}^{s} u_r y_{rj} - u_0 - \sum_{i=1}^{m} w_i x_{ij} + d_j - \beta_j = 0, \quad j = 1, \dots, n \qquad (7) \\
& \sum_{j=1}^{n} d_j = n - 1, \\
& 0 \le \beta_j \le 1, \quad d_j \epsilon \{0,1\}, \quad j = 1, \dots, n, \\
& w_i \ge \varepsilon^*, \; i = 1, \dots, m, \\
& u_r \ge \varepsilon^*, \; r = 1, \dots, s, \\
& u_0 \in U_0
\end{aligned}
$$

The objective function in this model is selected arbitrary as $(0 < \varepsilon^* < \infty)$ to resolve the infeasibility of model (6) to have a finite optimal solution.

Also, if $\varepsilon'^{*} = \varepsilon^{*} = \min\left(\frac{1}{\sum_{i=1}^{m} x_{ij}}\right); \; j = 1,\ldots,n$, the model (6) becomes feasible and vice versa. In this case the solution of model (6) and (7) is same.

By using the algorithm which presented by Foroughi [7], set of ME could be obtained in such a way that discriminate all of efficient DMUs through their efficiency with respect to different values of ε.

To select a single most efficient DMU from set ME, the following model was proposed

$$\begin{aligned}
&Max\,\alpha \\
&s.t. \\
&\sum_{i=1}^{m} w_i x_{ij} \le 1, \quad j = 1,\ldots,n, \\
&\sum_{r=1}^{s} u_r y_{rj} - u_0 - \sum_{i=1}^{m} w_i x_{ij} + d_j - \beta_j = 0, \qquad j \in ME, \\
&\sum_{r=1}^{s} u_r y_{rj} - u_0 - \sum_{i=1}^{m} w_i x_{ij} \le 0, \qquad j \notin ME, \\
&\alpha - \beta_j \le 0, \quad j \in ME, \\
&\sum_{j=1}^{n} d_j = |ME| - 1, \\
&0 \le \beta_j \le 1, \quad d_j \in \{0,1\}, \quad j \in ME, \\
&w_i \ge \bar{\varepsilon}, \quad i = 1,\ldots,m, \\
&u_r \ge \bar{\varepsilon}, \quad r = 1,\ldots,s, \\
&u_0 \in U_0.
\end{aligned} \tag{8}$$

This model provides a method to find the most efficient DMU by means of more discrimination with different return to scale such as VRS, CRS, NIRS, NDRS.

Although model (7) tries to improve previous models in order to solve their problem, it suffers from the problem that d_j may have an alternative optimal solution which introduces a set of most efficient DMUs instead of a single one. Model (8) then tries to improve model (7) further by proposing an algorithm, however, different sets of weights may arise in this model in such a way that each one introduces different ranking. Thus, it could not solve the problem completely.

Indeed, model (8) introduced a unit, as the most efficient, that could be realized to be target for many DMUs. However, the most efficient unit should have a full ranking, or in other words, it should have less threatening units in order to develop PPS more. In this regard, it is better to call this unit as the most effective DMU instead of the so-called most efficient.

In order to find the most efficient DMU, we propose a model that tries to solve the problem of previous models in the next section.

3 Proposed model

The proposed model tries to find the most efficient unit by means of hyperplanes.

More precisely, the desired hyperplane would be dominant on T_v and also the unit, which is determined by this hyperplane, must have a maximum distance from the other units. The model is defined as:

$$\begin{aligned}
& Max\, \alpha_1 \sum_j t_j + \alpha_2 \delta + \alpha_3 \sum_{j=1}^{n} \left(\sum_{r=1}^{s} u_r y_{rj} - u_0 \right) \\
& s.t. \\
& \sum_{i=1}^{m} v_i x_{ij} \le 1 \qquad j = 1, \ldots, n \\
& \sum_{r=1}^{s} u_r y_{rj} - u_o - \sum_{i=1}^{m} v_i x_{ij} + t_j = o \qquad j = 1, \ldots, n \quad (*) \\
& u \ge 1\delta \qquad v \ge 1\delta \\
& \sum_{j=1}^{n} y_j = n - 1 \\
& y_j \in \{o, 1\} \qquad j = 1, \ldots, n \\
& \varepsilon y_j \le t_j \le M y_j \qquad j = 1, \ldots, n
\end{aligned} \tag{9}$$

where α_1, α_2 and α_3 are positive parameters that show the importance of two objective functions to each other.

Note that maximizing δ brings u and v close to each other. In this model, getting closer the weights and finding the most efficient unit which has the largest distance from the other DMUs, is required. To achieve this purpose, we introduced t_j as a positive multiple of the distance of DMU_j from hyperplane. Now objective function is defined in such a way to increase distance of the most efficient DMU from the others, it tries to find a hyperplane which passing through only one efficient unit, thus the unit which is on the hyperplane is extreme point.

Constraint $\varepsilon y_j \le t_j \le M y_j$ guaranties if $y_j > 0$ then correspond slack (t_j) should be positive and it shows that constraint (*) is a hyperplane which does not pass through the corresponding DMU. Also, if $y_j = 0$, then $t_j = 0$, so constraint (*) is dominant and passes through DMU_j.

Constraint $\sum y_j = n - 1$, shows that only one Y is equal to zero and others equal to unity. In this case, the hyperplane passing through the unit which the associated Y is equal to zero and does not pass through the others. Thus, in this model, definitely, there is a hyperplane which passes through an efficient unit and necessarily has the largest distance from the other DMUs.

Theorem: Model (9) is feasible and its optimal objective value is finite.

Proof . Suppose that DMU_1 is an extreme efficient point on T_v (it is clear that, at least, there is an extreme efficient unit in T_v). Then, BCC-multiple form has a hyperplane which u and v are strictly positive.

Since DMU_1 is an extreme point, the only optimal solution for the BCC model then will be:

$$\theta^* = 1, \quad \lambda^* = e_1, \quad s^{+^*} = o, \quad s^{-^*} = o$$

Envelopment and multiple forms, which are dual, have an optimal solution which satisfies the strong condition of complementary slackness. Suppose that $\left(\lambda^*,\theta^*,s^{-^*},s^{+^*}\right)$ and $\left(u^*,v^*,u_o^*,t_*\right)$ are optimal solutions of envelopment and multiple forms. Since DMU_1 is an extreme point, thus envelopment form has only one optimal solution. Therefore, according to strong condition of complementary slackness:

$$\exists\left(u^*,v^*,u_o^*,t_*\right);$$

$$\begin{cases}\lambda_j^*.t_j^*=o\\ \lambda_j^*+t_j^*>o\end{cases},\quad \begin{cases}v_i^*.s_i^{-^*}=o\\ v_i^*+s_i^{-^*}>o\end{cases},\quad \begin{cases}u_r^*.s_r^{+^*}=o\\ u_r^*+s_r^{+^*}>o\end{cases}$$

we have: $\lambda_1^*>o\rightarrow t_1^*>o$ also $\lambda_2^*=\ldots=\lambda_n^*=o\rightarrow t_2^*,\ldots,t_n^*>o$, so $y_j\in\{0,1\}$ and $\sum_{j=1}^{n}y_j=n-1$, the model then gives $y_2^*=\ldots=y_n^*=1$. Since if all slacks are equal to zero, then u and v are strictly positive, so we can substitute $\delta^*=\min\{u_1^*,\ldots,u_s^*,v_1^*,\ldots,v_m^*\}$.

By first constraint ($\sum_{i=1}^{m}v_ix_{ij}\le 1$) of model we know $v^*x_1=1$ and for all the j, $VX_j=\alpha_j$ ($V>0$ & $X_j>0\rightarrow\alpha_j>0$) by dividing both side of above equation to α_j, we have $\frac{v}{\alpha_j}.X_j=1$ and $\left(\frac{X_j}{\alpha}=\bar{X}_j\right)$.

Defining $\bar{\alpha}=\mathrm{Max}\{\alpha_2,\ldots,\alpha_n\}$ and changing variables as $\left(\frac{u^*}{\bar{\alpha}},\frac{v^*}{\bar{\alpha}},\frac{u_o^*}{\bar{\alpha}},\frac{t^*}{\bar{\alpha}}\right)$ gives $\frac{v^*}{\bar{\alpha}}.X_j=\frac{\alpha_j}{\bar{\alpha}}\le 1;\ \forall j$, which shows the first constraint is satisfied. Imposing the defined variables, it is easy to show that the second constraint is also satisfied as $\frac{U^*Y_j-u_o^*-V^*X_j+t_j^*}{\bar{\alpha}}=0$. Dividing the third constraint into $\bar{\alpha}$ gives $\frac{U^*\ge 1\delta^*}{\bar{\alpha}}$ and $\frac{V^*\ge 1\delta^*}{\bar{\alpha}}\Rightarrow\delta^*\rightarrow\frac{\delta^*}{\bar{\alpha}}$. The constraints $\sum_{j=1}^{n}y_j=n-1$ and $y_j\in\{0,1\}$ will be substituted without changing. Since ε is very small and M is so large, it would be no problem, for the last constraint $\frac{\varepsilon Y_j}{\bar{\alpha}}\le\frac{t_j^*}{\bar{\alpha}}\le\frac{MY_j}{\bar{\alpha}}$ and $\xrightarrow{\frac{\varepsilon}{\bar{\alpha}}=\bar{\varepsilon}}$ $\bar{\varepsilon}Y_j\le\frac{t_j^*}{\bar{\alpha}}\le\bar{M}Y_j$. Therefore, the model is feasible.

On the other hand, the constraints $v\ge 1\delta$ and $\sum v_ix_{ij}\le 1$, gives a finite value for δ as $\delta\le\frac{1}{\sum x_{ij}}$. The expression $\alpha_1\sum_j t_j+\alpha_2\delta$ is finite since t_j is finite through the constraint $\varepsilon y_j\le t_j\le My_j$. Therefore, the proposed model has an optimal finite solution.

4 Numerical example

In this section, we use the real data of nineteen facility layout designs (FLDs) that are investigated by Ertay et al. [8]. Each FLD consumes two inputs, cost and adjacency score to produce shape ratio, flexibility, quality and hand-carry utility as four outputs. These data are shown in Table 1.

As we have mentioned previously, the proposed model could select a DMU as the most efficient one by means of the common set of weights and hyperplane. The efficiency score is calculated for each FLD and the results are summarized in Table 2.

Table 1 Inputs and outputs of FLDs

DMU	Inputs		outputs			
	Cost($)	adjacency	Shape ratio	flexibility	quality	Hand-carry utility
1	20309.56	6405.00	0.4697	0.0113	0.0410	30.89
2	20411.22	5393.00	0.4380	0.0337	0.0484	31.34
3	20280.28	5294.00	0.4392	0.0308	0.0653	30.26
4	20053.20	4450.00	0.3776	0.0245	0.0638	28.03
5	19998.75	4370.00	0.3526	0.0856	0.0484	25.43
6	20193.68	4393.00	0.3674	0.0717	0.0361	29.11
7	19779.73	2862.00	0.2854	0.0245	0.0846	25.29
8	19831.00	5473.00	0.4398	0.0113	0.0125	24.80
9	19608.43	5161.00	0.2868	0.0674	0.0724	24.45
10	20038.10	6078.00	0.6624	0.0856	0.0653	26.45
11	20330.68	4516.00	0.3437	0.0856	0.0638	29.46
12	20155.09	3702.00	0.3526	0.0856	0.0846	28.07
13	19641.86	5726.00	0.2690	0.0337	0.0361	24.58
14	20575.67	4639.00	0.3441	0.0856	0.0638	32.20
15	20687.50	5646.00	0.4326	0.0337	0.0452	33.21
16	20779.75	5507.00	0.3312	0.0856	0.0653	33.60
17	19853.38	3912.00	0.2847	0.0245	0.0638	31.29
18	19853.38	5974.00	0.4398	0.0337	0.0179	25.12
19	20335.00	17402.0	0.4421	0.0856	0.0217	30.02

Table 2 Efficiency scores of FLDs

Rank	DMU	Efficiency score
1	17	1.0000
2	9	0.9989
3	13	0.9965
4	7	0.9958
5	8	0.9878
6	18	0.9864
7	4	0.9839
8	1	0.9828
9	5	0.9826
10	12	0.9804
11	10	0.9793
12	6	0.9790

13	3	0.9751
14	11	0.9728
15	2	0.9704
16	14	0.9653
17	15	0.9601
18	16	0.9567
19	19	0.9512

As we could see, model (9) selects DMU_{17} as the most efficient unit by calculating the efficiency score for nineteen units and ranking them and the other DMUs are ranked respectively.

5. Conclusions

Finding the most efficient DMU in DEA is an important issue for decision makers. In this paper, we first briefly reviewed some models and their related problems which are suggested in this direction. We then proposed a model that determines the most efficient DMU. For this purpose, we first considered common set of weights for all DMUs and we then found a hyperplane which cross through only one unit. In contrast to the previous models that may have an alternative optimal solution, our proposed model selects a single unit, with large distance from the other units, as the most efficient DMU. We have also presented a numerical example with real data in order to justify the model.

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