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A new solving approach for fuzzy flexible programming problem in uncertainty conditions

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Abstract Modeling and solving real world problems is one of the most important issues in optimization problems. In this paper, we present an approach to solve Fuzzy Interval Flexible Linear Programming (FIFLP) problems that simultaneously have the interval ambiguity in the matrix of coefficients. In the first step, using the interval problem solving techniques; we transform the fuzzy interval flexible problem into two optimistic and pessimistic sub-problems. Then, in the second step, using a multi-parametric approach, we solve two fuzzy flexible sub- problems and finally the results are investigated with numerical example.

Keyword: Interval Linear Programming, Fuzzy Interval Flexible Linear Programming, $\bar{\alpha}$ –efficiency, $\bar{\alpha}$ –feasibility, Multi-parametric Approach.

1 Introduction

Fuzzy Linear Programming (FLP) problems allow working with imprecise data and constraints, leading to more realistic models. They have often been used for solving a wide variety of problems in sciences and engineering. Fuzzy mathematical programming has been researched by a number of authors. One of the earliest works on fuzzy mathematics programming problems was presented by Tanaka et al. [16] based on the fuzzy decision framework of Bellman and Zadeh [5]. In the literature, FLP has been classified into different categories, depending on how imprecise parameters are modeled by subjective preference-based membership functions or possibility distributions. Based on possible combinations of the fuzziness of the constraints matrix, resources vector, the cost coefficients and the objective function, Lai and Hwang [10] classified FLP problems into the following five general groups:

- **FLP of type-I**: The FLP problems with a fuzzy resources vector or fuzzy inequalityconstraints belong to this group.
- **FLP of type-II**: The FLP problems with a fuzzy resources vector (fuzzy inequality constraints) and a fuzzy objective function belong to this group.
- **FLP of type-III:** The FLP problems with fuzzy cost coefficients belong to this group.
- **FLP of type-IV**: The FLP problems with fuzzy cost coefficients, a fuzzy constraints matrix and a fuzzy resources vector (fuzzy inequality constraints) belong to this group.

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• **FLP of type-V:** The FLP problems with a fuzzy objective function, a fuzzy constraints matrix and a fuzzy resources vector (fuzzy inequality constraints) belong to this group.

Tanaka and Asai [16] proposed a possibilistic LP formulation where the coefficients of the decision variables were crisp while the decision variables were fuzzy numbers. Verdegay [17] proved that the optimal solution of an FLP of type-I can be found by the use of solving an equivalent crisp parametric LP problem assuming that the objective function is crisp. Werners [18, 19] suggested that the objective function should be fuzzy because of fuzzy inequality constraints and computed the lower and upper bounds of the optimal values by solving two crisp LP problems. Then, using the max-min operator of Bellman and Zadeh [5], he proposed a non-symmetric model for finding the solution of the FLP problem of type-I that satisfies the constraints and objective with the maximum degree. But the solution obtained by the max-min operator may not be efficient since the situation in the model proposed by Werners has multiple optimal solutions. To overcome this shortcoming, Guu and Wu [8] proposed a two-phase approach for solving the FLP problem of type-I that not only pursues the highest membership degree in the objective, but also pursues a better utilization of each constrained resource. After that, Nasseri and et al. in [11] introduced an equivalent fuzzy linear model for the flexible linear programming problems and proposed a fuzzy primal simplex algorithm to solve these problems. Recently, Attari and Nasseri [4] introduced a concept of feasibility and efficiency of solution for the fuzzy mathematical programming problems. The suggested algorithm needs to solve two classical associated linear programming problems to achieve an optimal flexible solution.

In the other hand, Interval linear programming problem which is a branch of linear programming was introduced originally by Ben and Rober's [6] which data are in the form of interval numbers, then the problem is an interval linear programming problem. They presented the first linear programming model for interval constraints for the first time. Subsequently, Huang and Moore introduced a new linear programming model in which all the parameters and variables were interval [9]. Generally, the solving method in these cases is the application of concepts that can turn the interval problem into problems with ordinary coefficients. These solving model transforms an interval linear programming problem into two sub-problems, one of which is the largest region with the best optimal value, and the other one obtains the smallest region with the worst optimal amount[3,7,15]. Many author research in interval problem with several objective function [13]. Recently, Mishmast Nehi & Allahdadi [1, 2] modified and improved the Tong method, which was unable to get optimal response on some issue. In this paper, we are going to suggest a new generalized model entitled "Fuzzy Flexible Interval Linear Programming (FFILP)" problem and using a multi-parametric approach propose a new two-phase method to solve the original FFILP problem suitably.

The rest of the paper is structured as follows: In section 2 we introduced the basic notions of interval linear programming and a Fuzzy Flexible Interval Linear programming (FFILP) problem. An application of the methods is described in fuzzy linear programming problems which contains interval numbers in the coefficients matrix in section 3. In Section 4, we present a numerical example for the proposed method. Section 5concluded the paper.

2. Preliminaries

In this section, some basic definitions, arithmetic operations and theorem that used in following are presented [1, 5].

Definition 1. Given $\underline{x}\, and\, \overline{x} \in \mathbb{R}$ such that $\underline{x} \leq \overline{x}$, we define a closed interval $x = [\underline{x}, \overline{x}]$ as the set $\{x \in \mathbb{R} : \underline{x} \leq x \leq \overline{x}\}$.

The values $\underline{x}$ *and* $\overline{x}$ are called the lower bound and upper bound of the interval x, respectively.

Definition 2. An interval $[\underline{x}, \overline{x}]$ with $\underline{x} = \overline{x}$ is said to be degenerate.

Since a degenerate interval $[\underline{x}, \overline{x}]$ only contains a single number, it is often identified with the number x itself, therefore it holds that $x = [x, x]$.

Definition 3. Given two matrices $\underline{A}\, and\, \overline{A} \in \mathbb{R}^{m\times n}$ such that $\underline{A} \leq \overline{A}$, we define a real interval matrix $A = [\underline{A}, \overline{A}]$ as the set $\{A \in \mathbb{R}^{m\times n} : \underline{A} \leq A \leq \overline{A}\}$. The matrices $\underline{A}\, and\, \overline{A}$ are called the lower bound and the upper bound of the interval matrix A, respectively.

The radius and center of A are $A_\Delta = \frac{1}{2}(\overline{A} - \underline{A})$ and $A_C = \frac{1}{2}(\overline{A} + \underline{A})$, respectively. Thus, $A = [\underline{A}, \overline{A}] = [A_C - A_\Delta, A_C + A]$.

An interval vector I is introduced as the set $\{I, \underline{I} \leq I \leq \overline{I}\}$, where $\underline{I}\, and\, \overline{I} \in \mathbb{R}^n$ are crisp vector[11].

Definition 4. A general form of the Interval Linear Programming (ILP) model is defined as follows:

$$\begin{aligned} & max \quad Z^{\pm} = \sum_{j=1}^{n} c_j^{\pm} x_j^{\pm} \\ & s.t. \qquad \sum_{j=1}^{n} a_{ij}^{\pm} x_j^{\pm} \leq b_i^{\pm}, \quad i = 1, 2, \ldots, m, \\ & \qquad\qquad x_j^{\pm} \geq 0, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

where $c_j^{\pm} \epsilon [c_j^-, c_j^+]$, $a_{ij}^{\pm} \epsilon [a_{ij}^-, a_{ij}^+]$ and $b_i^{\pm} \epsilon [b_i^-, b_i^+]$ are interval numbers and $x_j \epsilon [x_j^-, x_j^+]$ is an n-dimensional interval decision vector.

Theorem 1. In the ILP model (1), the largest and smallest feasible regions are $\sum_{j=1}^{n} a_{ij}{}^{+} x_j \leq b_i^-$, $i = 1, 2, \ldots, m$, $x_j \geq 0, j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} a_{ij}{}^{-} x_j \leq b_i^+$, $\forall i$, $x_j \geq 0, j = 1, 2, \ldots, n$, respectively [15].

There are several methods for solving interval linear programming problems, these methods for solving linear interval programming problems in such a way that in general the linear programming problem with interval parameters turns into two optimistic and pessimistic linear programming models where their solutions are the optimal interval of the main problem. This method examines the answers to the linear programming problems derived from the standard form [2, 15].

We transform the ILP problem (1) into pessimistic and optimistic sub-problems, which are summarized as follow [15]:

The pessimistic sub-problem:

$$\begin{aligned} max \quad & Z^- = \sum_{j=1}^{n} c_j^- x_i, \\ s.t. \quad & \sum_{j=1}^{n} a_{ij}^+ x_i \le b_i^-, \quad i = 1, 2, \ldots, m, \\ & x_j \ge 0, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{2}$$

The optimistic sub-problem:

$$\begin{aligned} max \quad & Z^+ = \sum_{j=1}^{n} c_j^+ x_i, \\ s.t. \quad & \sum_{j=1}^{n} a_{ij}^- x_i \le b_i^+, \quad i = 1, 2, \ldots, m, \\ & x_j \ge 0, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{3}$$

The optimal solutions to sub-problems (2), (3) is in box form as follows: $x^{\pm} = \left(x_1^{\pm}, x_2^{\pm}, ..., x_n^{\pm}\right)$, where for all $j = 1, 2, \ldots, n$, $x_j^{\pm} = \left[x_j^-, x_j^+\right]$. This box is the solution area of the tong method.

Theorem 2. In solving process of ILP model, if Z^* is the optimal objective value of model (1), and Z^{-*} and Z^{+*} are the optimal objective values of the model (2) and model (3), respectively, then $Z^* \in \left[Z^{-*}, Z^{+*}\right]$.

Definition 5: We consider a case where the decision maker assumes that there is a certain tolerance in the fulfillment of constraints. In other word, a certain degree of violation is allowed and this is created by the decision makers. The general form of the FFLP problems with fuzzy resources can be formulated as follows [9]:

$$\begin{aligned} max \quad & z = f\left(x, c\right) = \sum_{j=1}^{n} c_j x_j, \\ s.t. \quad & \mathrm{g}_i\left(x\right) = \sum_{j=1}^{n} a_{ij} x_j \preceq b_i, \ i = 1, 2, \ldots, m, \\ & x_j \ge 0, \quad j = 1, 2, ..., n. \end{aligned} \tag{4}$$

In model (4), "$\preceq$" is called " fuzzy less than or equal to" and it is assumed that the tolerance p_i for each constraint is given. This means that the decision maker can accept a violation of each constraint up to degree p_i. In this case, constraints $\mathrm{g}_i\left(x\right) \preceq b_i$, are equivalent to $\mathrm{g}_i\left(x\right) \le b_i + \theta p_i$, ($i$=1,2,…,m), where $\theta \in [0,1]$.

The problem (4) can be equivalently considered as the following fuzzy inequality constraints [6]:

$$max \quad z = f\left(x, c\right) = \sum_{j=1}^{n} c_j x_j,$$

$$s.t. \quad g_i(x) = \sum_{j=1}^{n} a_{ij} x_j \leq \tilde{b}_i,\ i = 1,2,\ldots,m, \tag{5}$$
$$x_j \geq 0, \quad j = 1,2,\ldots,n.$$

Definition 6. The membership function of the i- th constraints denoted as follows:

$$\mu_i(g_i(x)) = \begin{cases} 1, & g_i(x) \leq b_i, \\ 1-(g_i(x)-b_i)/p_i & b_i \leq g_i(x) \leq b_i + p_i, \\ 0 & g_i(x) \geq b_i + p_i \end{cases} \tag{6}$$

Now, by substituting membership function (6) into problem (5), the following crisp parametric LP problem is achieved:

$$max \quad z = f(x,c) = \sum_{j=1}^{n} c_j x_j,$$
$$s.t. \quad g_i(x) = (Ax)_i - b_i \leq (1-\alpha_i) p_i,\ \ i = 1,2,\ldots,m, \tag{7}$$
$$x_j \geq 0,\ \ \alpha_i \in (0,1], \quad j = 1,2,\ldots,n.$$

Definition 7: Let $\bar{\alpha} = (\alpha_1,\ldots,\alpha_m) \in (0,1]^m$ be a vector, and $X_{\bar{\alpha}} = \left\{ x \in R^n \setminus x \geq 0, \mu_i \left\{ g_i(x,a_i) \preceq 0 \right\} \geq \alpha_i,\ i = 1,2,\ldots,m. \right\}$.

Then, a vector $x \in X_{\bar{\alpha}}$ is called an $\bar{\alpha}$ - feasible solution of model (5).

Following proposition enables us to define feasible set of model (5) as an intersection of all α -cuts corresponding to fuzzy constraints.

Proposition 1: Let $\bar{\alpha} = (\alpha_1,\ldots,\alpha_m) \in (0,1]^m$, then $X_{\bar{\alpha}} = \bigcap_{i=1}^{m} X_{\alpha_i}^{i}$, where $X_{\alpha_i}^{i} = \left\{ x \in R^n \setminus x \geq 0, \mu_i \left\{ g_i(x,a_i) \preceq 0 \right\} \geq \alpha_i \right\}$, for $i \in I = \{1,\ldots,m\}$ (Namely, X_{α}^{i} is the α -cuts of the i –th constraint).

Proof: The proof is straightforward. ■

Proposition 2: Let $\overline{\alpha'} = (\alpha'_1,\ldots,\alpha'_m)$ and $\overline{\alpha''} = (\alpha''_1,\ldots,\alpha''_m)$, where $\alpha_i^{'} \leq \alpha_i^{''}$ for all i. Then $\overline{\alpha''}$ -feasibility of x implies the $\overline{\alpha'}$ -feasibility of it.

Proof: The proof is straightforward. ■

Definition 8: Let "$\preceq$" be a fuzzy extension of relation "$\leq$" and a solution $X = (x_1,\ldots,x_m)^T \in \mathbb{R}^n$ be an $\bar{\alpha}$ – feasible to problem (5), where $\bar{\alpha} = (\alpha_1,\ldots,\alpha_m) \in (0,1]^m$ and let $f(x,c)$ be an objective function in the form of maximization. Then, $X = (x_1,\ldots,x_n)$, where $x_j \in R^n$ is an $\bar{\alpha}$ – efficient solution to problem(5), if there is no $x' \in X_{\bar{\alpha}}$ so that $f(x,c) < f(x',c)$.

Clearly, any $\bar{\alpha}$ – efficient solution to the FFLP is an $\bar{\alpha}$ – feasible solution to the FFLP with some additional properties.

3. Fuzzy Flexible Interval Linear Programming

Let us consider a general model of Fuzzy Flexible Interval Linear Programming (FFILP) model as follows:

$$\begin{aligned} & max \quad z = f\left(x,c\right) = \sum_{j=1}^{n} c_j x_j, \\ & s.t. \quad \sum_{j=1}^{n} a^{\pm}{}_{ij} x_j \preceq b_i,\ i = 1,2,\ldots,m, \\ & x_j \geq 0, \quad j = 1,2,...,n. \end{aligned} \tag{8}$$

where $x = \left(x_1, x_2, ..., x_n\right)^T$ is a real vector of decision variables, and interval where $\tilde{c}_j$ is a fuzzy number that is the objective coefficients. $a^{\pm}{}_{ij}$ shows an interval coefficient matrix as $A = \left[a^{-}{}_{ij}, a^{+}{}_{ij}\right]$, where A is a $m \times n$- dimensional matrix of interval technical coefficients. And, objective functions and constraints where $i \in \{1,...,m\}$ possess continuous property up to the second derivatives. Also, "$\preceq$" denote a fuzzy extension of "$\leq$" on $\mathbb{R}$ which is used to compare the left and right side of fuzzy constraints [12,14]. Unfortunately, the problem (12) is not well-defined because of: in first step, we can't use linear programming techniques to solving linear problems with interval coefficients matrix and obtain a feasible area of problem. In second step, we can't achieve a crisp feasible set of the constraints $\mathrm{g}_i\left(x, a_i\right) \preceq 0$, $i \in \{1,...,m\}$.

To solving the above mentioned problem with two various types of uncertainty associated with the coefficients matrix and flexible constraints, we present a two- step approach.

In first step, by use of interval techniques and methods that proved to solving a linear programming problem with interval coefficients. This methods in general the linear programming problem with interval parameters turns into two optimistic and pessimistic linear programming models where their solutions are the optimal interval of the main problem [2, 15].

Now, we transformed the ILP problem (8) into pessimistic and optimistic sub-problems, which are summarized as follow [3, 15]:

The pessimistic sub-problem:

$$\begin{aligned} & max \quad Z^{-} = f\left(x,c\right) = \sum_{j=1}^{n} c_j x_j, \\ & s.t. \quad \sum_{j=1}^{n} a_{ij}^{+} x_i \preceq b_i, \quad i = 1,2,\ \ldots,\ m, \\ & x_j \geq 0, \quad j = 1,2,\ \ldots,\ n. \end{aligned} \tag{9}$$

The optimistic sub-problem:

$$\begin{aligned} & max \quad Z^{+} = f\left(x,c\right) = \sum_{j=1}^{n} c_j x_j, \\ & s.t. \quad \sum_{j=1}^{n} a_{ij}^{-} x_i \preceq b_i, \quad i = 1,2,\ \ldots,\ m, \\ & x_j \geq 0, \quad j = 1,2,\ \ldots,\ n. \end{aligned} \tag{10}$$

The optimal solutions to sub-problems (9) and (10) is in box form as follows: $x^{\pm} = \left(x_1^{\pm}, x_2^{\pm}, ..., x_n^{\pm}\right)$, where for all $j = 1,2,\ \ldots,\ n$, $x_j^{\pm} = \left[x_j^{-},\ x_j^{+}\right]$. This box is the solution area of the tong method.

In final step, in order to find maximum efficient solution, i.e. an $\bar{\alpha}$ - efficient solution with $\tilde{\alpha} \geq \alpha$, $i = 1, 2, \ldots, m$, we perform the following two phase approach. To express this two-phase approach to the above problem, let us consider the problem (9) and implement a two-phase approach for this sub-problem, and then, with the resumption of the approach discussed below, we solve the second problem. In the two phase approach, Eq (6) is solved in Phase I, while in Phase II, a solution is obtained which has higher satisfaction degrees than the previous solution. Thus by using this two Phase approach, we achieve a better utilization of available resources. Further the solution resulting by this approach is always an $\bar{\alpha}$ - efficient solution. Let us consider the Definition and substituting in the problem (9) achieve the parametric linear programming that solved by linear techniques.

$$\begin{aligned} max \quad & Z^- = f(x,c) = \sum_{j=1}^{n} c_j x_j, \\ s.t. \quad & \sum_{j=1}^{n} a_{ij}^+ x_j \leq b_i + (1-\alpha_i)\, p_i, \quad i = 1, 2, \ldots, m, \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{11}$$

Let us call the problem (11) as Phase I problem.

Theorem 3.2: Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (0,1]^m$ and also $X^* = \left(x_1^*, \ldots, x_m^*\right)^T$, where $x_j^* \geq 0$, $j \epsilon \{1, 2, \ldots, n\}$ be an $\bar{\alpha}$ – feasible solution to problem (9). Then, $x^* \in \mathbb{R}^n$ is an $\bar{\alpha}$ – efficient optimal solution to problem (9), where assumed that the objective function is in the type of maximization, if and only if the decision making x^* is an optimal solution of problem (11), where p_i is the predefined maximum tolerance.

Proof: Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (0,1]^m$ and $X^* = \left(x_1^*, \ldots, x_m^*\right)^T$, so that $x_j^* \geq 0$, $j \epsilon \{1, 2, \ldots, n\}$, be an $\bar{\alpha}$ – efficient solution to problem (11). With regard to Definition 7 and equation (6) concluded that x^* is feasible to problem (9). Because of $\mu_i \left\{ \mathrm{g}_i (x, a_i) \preceq 0 \right\} \geq \alpha_i$ or equivalently $\sum_{j=1}^{n} a_{ij}^+ x^*{}_j \leq b_i + (1-\alpha_i)\, p_i$ for $i \in \{1, \ldots, m\}$ *and* $j = 1, 2, \ldots, n.$

Also, by Definition 8, there is no $x' \in X_{\bar{\alpha}}$ such that $f(x,c) < f(x',c)$, it means that x^* is optimal to problem(13). In other hand, if x^* is an optimal solution of problem (9), obviously, x^* is an $\bar{\alpha}$ – feasible solution to problem (9) and hence, the optimality of x^* implies that the $\bar{\alpha}$ – efficiency of x^* .

Also, in theorem 3.2, we discussed the method for obtaining the $\bar{\alpha}$ -efficiency solution of fuzzy mathematical programming problem. If the result of problem (11) has only one optimal answer, then this answer is an $\bar{\alpha}$ -efficient solution given for the fuzzy problem. When problem (11) has several optimal answers, use two phase approach in order to find the maximum effective response $\bar{\alpha}$ -efficient solution with $\bar{\alpha} \geq \alpha$ and i = 1,2, ..., m. In the two-stage approach, problem (11) is solved in the first phase. Then, in Phase II, an answer is obtained that has a higher degree of validity than the previous one. So, using this two-step approach, we will achieve a better productivity of existing resources and make the decision-maker more convincing and more satisfying.

Let $\overline{\alpha^0} = \left(\alpha^0{}_1, \ldots, \alpha^0{}_m\right)$ and $\left(x^{-*}, f\left(x^{-*}, c\right)\right)$ be the optimal solution of pessimistic sub-problem of Phase I with $\overline{\alpha^0}$ degree of efficiency. Set $\alpha_i^* = \mu_i\left\{g_i\left(x^*, a_i\right) \preceq 0\right\} \geq \alpha_i^0,\ i = 1, \ldots, m.$ In Phase II, we solve the following problem,

$$\begin{aligned} & max \quad \sum_{i=1}^{m} \alpha_i \\ & s.t. \left(x^-, f\left(x^-, c\right)\right) \geq \left(x^{-*}, f\left(x^{-*}, c\right)\right) \qquad (12) \\ & \quad \sum_{j=1}^{n} a_{ij}^+ x_i \leq b_i + \left(1 - \alpha_i\right) p_i, \quad i = 1, 2, \ldots, m, \\ & \quad x_j \geq 0,\ \alpha_i^* \leq \alpha_i \leq 1, j = 1, 2, \ldots, n. \end{aligned}$$

Theorem 3.3: The optimal solution x^{**} to problem (12) is a maximum $\bar{\alpha}$ − efficient solution to problem (9).
Proof: The proof is straightforward. ■

3.1 The main steps of algorithm

Here, we are going to present the main steps of our approach for solving Fuzzy Flexible Interval Linear Programming (FFILP) problems which is defined in the problem (8).
Assumption 1: In problem (8), only the coefficient matrix is interval and another parameters in constraints is crisp.
Assumption 2. An approach to transform n interval linear programming problem in two linear programming sub- problem is given (see in [1, 2, 3, 7, 15])
Step1: By use of interval problems method, obtain two sub-problem (9) and (10) that coefficient matrix is crisp.
Step 2: given pessimistic sub- problem (9) and based on relation (4-7), the problem is rewritten as multi-parametric problem (11).
Step 3: Solve the above multi-parametric problem as Phase I and first obtain the optimal value of x^* and α^*, and then the optimal value of the objective function Z^*.
Step 4: Based on the optimal solution of problem in step 3 reformat the multi-parametric problem as the as (12)
Step 5: Consider second sub-problem (10) and go to step 2.

4. Numerical Example

A manufacturer of metal office equipment makes desk, chairs, cabinets and bookcases. The work is carried out in three departments: (1) metal stamping, (2) assembly, (3) finishing. The pertinent data are presented in the table below. Note that, the hours that per week available are increased by 300 hours for the stamping, 500 hours for the assembly and 170 hours for finishing.

Table1. Pertinent data

Department	Desk	Chair	Cabinet	Bookcase	Hours per week available
Stamping	3-15	1.5-8	2-12	2-12	800
Assembly	10-30	6-18	8-24	7-21	1200
Finishing	10-35	8-28	8-25	7-21	800
Selling price	175	95	145	130	

what should be the rate of production of each of the items be in order to maximize weekly profits if no restrictions are placed on the number of any item to be made.

The mathematical formulation of the above problem can be summarized as follows:

$$\begin{aligned} \max \quad & z = 175x_1 + 95x_2 + 145x_3 + 130x_4 \\ s.t. \quad & [3,15]\ x_1 + [1.5,8]\ x_2 + [2,12]x_3 + [2,12]x_4 \preceq 800, \\ & [10,30]x_1 + [6,18]x_2 + [8,24]x_3 + [7,21]x_4 \preceq 1200, \\ & [10,35]\ x_1 + [8,28]x_2 + [8,25]x_3 + [7,21]x_4 \preceq 800, \\ & x_j \ge 0, \qquad j = 1,2,3,4. \end{aligned} \tag{13}$$

Now, using the interval method that mentioned in this paper and transforming the above interval problem (13) into two crisp sub-problems. Firs, consider one of two sub-problem and with the member ship function (6), where $P_1 = 300$, $P_2 = 500$ and $P_3 = 170$ are predefined maximum tolerance from b_i, i=1,2,3. By substitute the above membership function (6) into each sub-problems obtains a multi-parametric problem as follow:

$$\begin{aligned} \max \quad & z^+ = 175x_1 + 95x_2 + 145x_3 + 130x_4 \\ s.t. \quad & 3x_1 + 1.5x_2 + 2x_3 + 2x_4 \le 800 + 300(1-\alpha_1), \\ & 10x_1 + 6x_2 + 8x_3 + 7x_4 \le 1200 + 500(1-\alpha_2), \\ & 10x_1 + 8x_2 + 8x_3 + 7x_4 \le 800 + 170(1-\alpha_3), \\ & x_j \ge 0, \qquad j = 1,2,3,4. \end{aligned} \tag{14}$$

Let $x^* = (0,0,0,126.4286)$ be (0.8, 0.5, 0.5)-efficient solution with $C^T x^* = 16435.71$ as an optimal value of problem. In Phase II of parametric approach we need to solve the following linear problem:

$$\begin{aligned} max \quad & \sum_{i=1}^{3} \alpha_i \\ s.t. \quad & 175x_1 + 95x_2 + 145x_3 + 130x_4 \ge 16435.71, \\ & 3x_1 + 1.5x_2 + 2x_3 + 2x_4 \le 800 + 300(1-\alpha_1), \\ & 10x_1 + 6x_2 + 8x_3 + 7x_4 \le 1200 + 500(1-\alpha_2), \\ & 10x_1 + 8x_2 + 8x_3 + 7x_4 \le 800 + 170(1-\alpha_3), \\ & 0.8 \le \alpha_1 \le 1, 0.5 \le \alpha_2 \le 1, 0.5 \le \alpha_3 \le 1, \\ & x_j \ge 0, \qquad j = 1,2,3. \end{aligned} \tag{15}$$

An optimal solution to the above problem is $x^{**} = (0,0,0,126.4285)$, also $C^T x^* = C^T x^{**} = 16435.71$ we have $\mu_1 = \mu_2 = 1, \mu_3 = 0.5$. Using the two phase approach, we can get an optimal solution x^* which not only achieves the optimal objective value but also give a higher value in μ_1, μ_2.

Now, we use all this steps to solve the second sub-problem. Finally, by solving the second sub-problem obtain that, if $x^* = (0,0,0,42.14286)$ be (0.8, 0.5, 0.5)-efficient solution with $C^T x^* = C^T x^{**} = 5478.571$, and $x^{**} = (0,0,0,42.14286)$ and $\mu_1 = \mu_2 = 1, \mu_3 = 0.5$. Finally, with regards to Theorem 2 optimal objective value of problem (13) is $Z^* = [5478.571, 16435.71]$.

5. Conclusion

In this paper, we investigated a linear programming problem with interval data and flexible constraints. We saw that the mentioned model is more adaptive with the practical situations. In particular, a new concept of $\bar{\alpha}$-feasibility and $\bar{\alpha}$-efficiency of solution in fuzzy flexible linear programming problems is introduce to propose a parametric approach for solving the original problem by solving two associated classical linear programming problems. This approach will be useful in obtaining flexible responses with a degree of satisfaction determined by the decision maker for fuzzy mathematical programming. Interval techniques help us to convert linear problem with interval uncertainty to crisp problems and by using of linear techniques solve them.

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