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More on equivalent formulation of implicit complementarity problem

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Abstract This article introduces a new way to express the implicit complementarity problem and shows that solving this new version is just as effective as solving the original problem. The authors also present another alternative version of the problem, which is based on using a strictly increasing function. Both approaches provide equivalent solutions, offering potentially more efficient or insightful ways to address the original problem.

Keyword: Implicit Complementarity Problem, Equivalent Form, Strictly Increasing Function.

1 Introduction

The implicit complementarity problem (ICP) is a generalization of the classical complementarity problem that arises in a variety of applications such as optimization, economic equilibrium models, and engineering systems. Given its complexity and wide applicability, researchers have explored various equivalent formulations to analyse, solve, and understand the problem. For details see, [1], [2] and [3]. Bensoussan et al. [4] presented the implicit complementarity problem (ICP) and it is a class of mathematical optimization problems that solves a system of nonlinear equations which includes both complementary conditions and equality or inequality constraints. ICPs can be formulated as follows:

Consider the matrix $\mathrm{A} \in \mathrm{R}^{n\times n}$ and the vector $b \in \mathrm{R}^n$, the implicit complementarity problem denoted as ICP(A, b, f) is to find the solution $r \in \mathrm{R}^n$ to the following system:

$$\mathrm{H}(r) = r - f(r) \geq 0, \qquad \mathrm{F}(r) = \mathrm{A}r + b \geq 0, \qquad \mathrm{H}(r)^T\,\mathrm{F}(r) = 0, \qquad (1.1)$$

where $f(r)$ is a mapping from R^n to R^n.

The study of equivalent forms of the implicit complementarity problem is crucial for advancing both theoretical understanding and practical solution methods. By transforming ICP into alternative formulations, researchers can leverage specialized techniques and tools, leading to more efficient and robust solutions. By leveraging these equivalent formulations,

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researchers have developed algorithms with improved convergence properties and computational efficiency.

One of the most popular techniques for developing fast and affordable iterative algorithms is the equivalent formulation of the linear complementarity problem (LCP) as an equation with the same solution. The LCP (A, *b*) is described in an analogous form and several iteration techniques are given by Bai in [2]. For more details on equivalent form of LCPs and related iteration methods see, [6], [7], [8], [9] and [10]. The concept of equivalent formulation has also been used effectively for other complementarity problems, like implicit complementarity problem [8] and [11] and horizontal linear complementarity problem [12].

Mangasarian offered an equivalent forms of LCP(A, *b*) in [13] and described as

$$r = (r - \omega\Omega(\mathrm{A}r + b))_+,$$

where $r_+ \in \mathrm{R}^n$, $(r_+)_i = max\{0, r_i\}$ and $\Omega \in \mathrm{R}^{n\times n}$ is a positive diagonal matrix. Motivated by the works of Mangasarian [13], we present an equivalent form of ICP.

The article is structured as follows: Section 2 introduces an equivalent formulation of the ICP and outlines conditions necessary for obtaining its solution. Section 3 presents the conclusions, summarizing the key insights and implications of the study.

2 Main Results

We start by outlining certain fundamental notations that will be utilized in this study. We take into account real matrices and vectors. R^n implies the n dimensional space of real entries. $r \in \mathrm{R}^n$ is a column vector and r_i implies i^{th} component of the vector $r \in \mathrm{R}^n$.

Now, we provide an equivalent expression of the implicit complementarity problem. The equivalence form of ICP(A, *b*, *f*) is

$$\mathrm{P}(r) = \mathrm{H}(r) - (\mathrm{H}(r) - (\mathrm{A}r + b))_+,$$

In the following result, we demonstrate that the equivalently formulation of ICP(A, b, f) has the same solution.

Theorem 2.1. Suppose $\mathrm{A} \in \mathrm{R}^{n\times n}$ and $b \in \mathrm{R}^n$. Then $r^* \in \mathrm{R}^n$ be the solution of ICP(A, *b*, *f*) if and only if $\mathrm{P}(r^*) = 0$, where $\mathrm{P} : \mathrm{R}^n \to \mathrm{R}^n$ is defined as

$$\mathrm{P}(r) = \mathrm{H}(r) - (\mathrm{H}(r) - (\mathrm{A}r + b))_+, \tag{2.1}$$

Proof. Suppose $\mathrm{P}(r^*) = 0$, it follows that $(\mathrm{H}(r^*) - (\mathrm{H}(r^*) - (\mathrm{A}r^* + b))_+) = 0$. This implies that

$$\mathrm{H}(r^*) = (\mathrm{H}(r^*) - (\mathrm{A}r^* + b))_+, \tag{2.2}$$

Component-wise, we consider two cases:

Case 1. when $\mathrm{H}_i(r^*) \geq (\mathrm{A}r^* + b)_i$, where $\mathrm{H}_i(r^*)$ denotes the i^{th} component of the $\mathrm{H}(r^*)$.

Then Equation (2.2) can be written as $\mathrm{H}_i(r^*) = \mathrm{H}_i(r^*) - (\mathrm{A}r^* + b)_i$.

It follows that, $\mathrm{F}_i(r^*) = (\mathrm{A}r^* + b)_i = 0$.

Case 2. when $\mathrm{H}_i(r^*) < (\mathrm{A}r^* + b)_i \Rightarrow ((\mathrm{A}r^* + b)_i - \mathrm{H}_i(r^*)) < 0$. Then, we get $\mathrm{H}_i(r^*) = 0$.

From case (1) and case (2), $\mathrm{H}_i(r^*)\mathrm{F}_i(r^*) = 0 \ \forall \ i \Rightarrow \mathrm{H}(r^*)^T \ \mathrm{F}(r^*) = 0$.

Conversely, let r^* be the solution of system (1.1). By complementary condition of ICP, either $\mathrm{H}_i(r^*) = 0$ or $\mathrm{F}_i(r^*) = 0 \ \forall \ i$.

Component-wise, we consider two cases:

Case 1. If $\mathrm{H}_i(r^*) = 0$ and $\mathrm{F}_i(r^*) > 0$, Equation (2.1) becomes

$$\mathrm{P}_i(r^*) = (-(-(\mathrm{A}r^* + b))_+) \Rightarrow \mathrm{P}_i(r^*) = 0,$$

Case 2. If $\mathrm{H}_i(r^*) > 0$ and $\mathrm{F}_i(r^*) = 0$, then $\mathrm{P}_i(r^*) = (\mathrm{H}_i(r^*) - (\mathrm{H}_i(r^*))_+) \Rightarrow \mathrm{P}_i(r^*) = 0$.

From case (1) and case (2), we get $\mathrm{P}_i(r^*) = 0 \ \forall \ i$. Then $\mathrm{P}(r^*) = 0$. □

Remark 2.1. Let $\mathrm{S}(r) = \mathrm{H}(r) - \mathrm{P}(r)$, then r^* is the solution of ICP(A, *b*, *f*) if and only if $\mathrm{S}(r^*) = \mathrm{H}(r^*)$.

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In the following result, we show that the ICP(A, b, f) can be equivalently formulated as an equation with $\Omega_1, \Omega_2 \in R^{n\times n}$ be two positive diagonal matrices whose solution remains unchanged.

Lemma 2.1. Suppose $A \in R^{n\times n}$ and $b \in R^n$. Let $\Omega_1, \Omega_2 \in R^{n\times n}$ be two positive diagonal matrices and define the mapping $\bar{P}(r) = (\Omega_1 H(r) - (\Omega_1 H(r) - \Omega_2(Ar + b))_+)$. Then r^* is the solution of ICP(A, b, f) if and only if $\bar{P}(r^*) = 0$.

Proof. Suppose $\bar{P}(r^*) = 0$, it follows that $(\Omega_1 H(r^*) - (\Omega_1 H(r^*) - \Omega_2(Ar^* + b))_+) = 0$.

Then we write

$$\Omega_1 H(r^*) = (\Omega_1 H(r^*) - \Omega_2(Ar^* + b))_+, \tag{2.3}$$

Component-wise, we consider two cases:

Case 1. when $(\Omega_1 H(r^*))_i \geq (\Omega_2(Ar^* + b))_i$, then Equation (2.2) becomes

$$(\Omega_1 H(r^*))_i = (\Omega_1 H(r^*))_i - (\Omega_2(Ar^* + b))_i,$$

Then, $(\Omega_2(Ar^* + b))_i = 0$. This implies $F_i(r^*) = (Ar^* + b)_i = 0$.

Case 2. when $(\Omega_1 H(r^*))_i < (\Omega_2(Ar^* + b))_i$, this implies that:

$((\Omega_2(Ar^* + b))_i - (\Omega_1 H(r^*))_i) < 0$

Then, we get $(\Omega_1 H(r^*))_i = 0 \Rightarrow H_i(r^*) = 0$.

From case (1) and case (2), we obtain $H_i(r^*)F_i(r^*) = 0\ \forall\ i$. Hence, $H(r^*)^T F(r^*) = 0$.

Conversely, let r^* be the solution of system (1.1). Then component-wise, we consider two cases:

Case 1. when $H_i(r^*) = 0$ and $F_i(r^*) > 0$ implies $\bar{P}i(r^*) = -(-(\Omega_2(Ar^* + b))_i)_+ \Rightarrow \bar{P}i(r^*) = 0$.

Case 2. when $H_i(r^*) > 0$ and $F_i(r^*) = 0$, then $\bar{P}i(r^*) = ((\Omega_1 H(r^*))_i - ((\Omega_1 H(r^*))_i)_+)$.

Thus, $\bar{P}i(r^*) = 0$.

From case (1) and case (2), we obtain $\bar{P}(r^*) = 0$. □

In the following result, we show that the ICP(A, b, f) can be equivalently formulated as an equation with any strictly increasing function such that $\delta(0) = 0$, whose solution must be same as the ICP(A, b, f).

Theorem 2.2. Suppose $A \in R^{n\times n}$ and $b \in R^n$. Let $\delta : R \to R$ be any strictly increasing function such that $\delta(0) = 0$ .Then r^* is the solution of ICP(A, b, f) if and only if $G(r^*) = 0$, G is the function from R^n to R^n, given as

$$G_i(r) = \delta(|(Ar + b)_i - H_i(x)|) - \delta((Ar + b)_i) - \delta(H_i(x)),\ i = 1, 2, \ldots, n, \tag{2.4}$$

Proof. For some i, let $H_i(r^*) < 0$. Then it follows that

$$0 > \delta(H_i(r^*)) = \delta(F_i(r^*) - H_i(r^*)) - \delta(F_i(r^*)) \geq -\delta(F_i(r^*)), \tag{2.5}$$

Thus, $F_i(r^*) > 0$ and $F_i(r^*) - H_i(r^*) > F_i(r^*) > 0$. This implies that

$$\delta(|F_i(r^*) - H_i(r)|) = \delta(F_i(r^*) - H_i(r^*)) > \delta(F_i(r)), \tag{2.6}$$

From inequalities (2.5) and Equation (2.6), we get $G(r^*) > 0$, this is the contradiction.

If $H_i(r^*) > 0$ and $F_i(r^*) > 0$ for some i. Then, we consider two possibilities:

When $H_i(r^*) > F_i(r^*)$, then $\delta(|F_i(r^*) - H_i(r^*)|) = \delta(H_i(r^*) - F_i(r^*)) < \delta(H_i(r^*))$. Therefore

$$\delta(|F_i(r^*) - H_i(r^*)|) - \delta(H_i(r^*)) < 0,$$

It follows that

$$G_i(r^*) < 0, \tag{2.7}$$

When $F_i(r^*) > H_i(r^*)$, then $\delta(|F_i(r^*) - H_i(r^*)|) = \delta(F_i(r^*) - H_i(r^*)) < \delta(F_i(r^*))$.

Hence $\delta(|F_i(r^*) - H_i(r^*)|) - \delta(F_i(r^*)) < 0$. This implies that

$$G_i(r^*) < 0, \tag{2.8}$$

From inequalities (2.7) and (2.8), we must have $G_i(r^*) < 0$, again a contradiction.

Therefore, r^* solves the ICP(A, b, f).

Conversely, let r^* be the solution of ICP(A, b, f), then either $H_i(r^*) = 0$ or $F_i(r^*) = 0\ \forall\ i$.

Suppose $H_i(r^*) = 0$ and $F_i(r^*) > 0$, then $G_i(r^*) = \delta(|F_i(r^*)|) - \delta(F_i(r^*))$.

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This implies that $G_i(r^*) = 0$.
When $F_i(r^*) = 0$ and $H_i(r^*) > 0$, Then $G_i(r^*) = \delta(|-H_i(r^*)|) - \delta(H_i(r^*))$.
This implies that $G_i(r^*) = 0 \ \forall \ i$. Therefore, $G(r^*) = 0$. □

3 Conclusion

In this article, we presented an equivalent formulation of the implicit complementarity problem. We demonstrated that the solution of this equivalent formulation is identical to the solution of the original implicit complementarity problem. Additionally, by utilizing a strictly increasing function δ, we provided another equivalent form of the implicit complementarity problem. The concept of equivalent formulations proves to be an effective approach for solving complementarity problems.

Availability of data and materials
Not applicable.

Contribution
All authors contributed equally.

Conflict of interest
The authors declare that no conflicts of interest exist.

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References

1. Ferris MC, Mangasarian OL, Pang JS (2011) Complementarity: Applications, Algorithms and Extensions. Springer, New York.
2. Lemke CE, Howson JT (1964) Equilibrium points of bimatrix games. SIAM J Appl Math 12:413-423.
3. Murty KG (1988) Linear Complementarity, Linear and Nonlinear Program- ming. Heldermann, Berlin.
4. Bensoussan A, Lions JL (1975) Nouvelles Methodes en Controle Impulsionnel, Appl Math Optim 1:289-312.
5. Bai ZZ (2010) Modulus based matrix splitting iteration methods for linear complementarity problems. Numer Linear Algebra Appl 17(6):917-933.
6. Ahn BH (1981) Solutions of nonsymmetric linear complementarity problems by iterative methods. J Opt Theory Appl 33:175-185.
7. Cottle RW, Pang JS, Stone RE (1992) The linear complementarity Problem. Academic Press London.
8. Hong JT, Li CL (2016) Modulus based matrix splitting iteration methods for a class of implicit complementarity problems. Numer Linear Algebra Appl 23:629-641.
9. Kumar B, Deepmala, Das AK (2023) Projected fixed point iterative method for large and sparse horizontal linear complementarity problem. Indian J Pure Appl Math. https://doi.org/10.1007/s13226-023-00403-4.
10. Noor MA (1988) Fixed point approach for complementarity problems. J Math Anal Appl 133:437-448.
11. Xie SL, Xu HR, Zeng JP (2016) Two step modulus-based matrix splitting iteration method for a class of nonlinear complementarity problems. Linear Algebra Appl 494:1-10.
12. Mezzadri F, Galligani E (2020) Modulus based matrix splitting methods for horizontal linear complementarity problems. Numer Alg 83:201-219.
13. Mangasarian O (1977) Solution of symmetric linear complementarity problems by iterative methods. J Optim Theory Appl 22:465-485.

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