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# A New fixed point model for linear complementarity problems

S. A. Edalatpanah*

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**Abstract** In this paper, we propose a new fixed point model for the linear complementarity problem (LCP). The equivalent of this model and LCP has been proved. The structure of this model is simple and can be applied to iterative methods for LCPs and also other complementarity problems.

**Keywords**: Linear Complementarity Problem, Fixed Point Model, Projected Model, Modulus Model.

## 1 Introduction

For a given real vector $q \in R^n$ and a given matrix $A \in R^{n\times n}$, the linear complementarity problem abbreviated as *LCP (A, q)*, consists in finding vectors $z \in R^n$ such that

$$\begin{cases} w = Az - q \\ z \geq 0, w \geq 0 \\ z^T w = 0 \end{cases} \tag{1}$$

where $z^T$ denotes the transpose of the vector $z$. Many problems in various scientific computing, economics and engineering areas can lead to the solution of *LCP* and its generalizations. For example, quadratic programming, Nash equilibrium point of a bimatrix game, nonlinear obstacle problems, invariant capital stock, optimal stopping, contact and structural mechanics, free boundary problem for journal bearings, traffic equilibriums, manufacturing systems, etc. For more details, see [1-3] and the references therein. Because of the wide applications, the research on the numerical methods for solving (1) has attracted much attention.

Numerous methods exist for solving the linear complementarity problems (LCPs). Numerical algorithms to solve Eq. (1) fall in two main classes, direct and iterative. Direct methods are those based on the process of pivoting, that is, exchanging the roles of dependent and independent variables (similar to basic and non-basic variables in a system of equations), while iterative methods are those which produce a (possibly infinite) sequence of iterates (trial solutions) which converge to a solution [1-3].

One of the oldest iterative methods related to the linear complementarity problem is due to Hildreth [4], who designed the procedure to solve a strictly convex quadratic program.

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*Corresponding Author. (✉)
**E-mail**: saedalatpanah@gmail.com (S.A. Edalatpanah)

**S. A. Edalatpanah**
Assistance Professor, Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran

Hildreth stated its Kuhn-Tucker conditions and used the nonsingularity of the Hessian matrix of the objective function to eliminate the primal variables. What remains after this operation is a linear complementarity problem in variables are Lagrange multipliers and the matrix $A$ is symmetric and positive semi-definite. A more general iterative method, attributed to Christopherson [5], has been analyzed and clarified by Cryer [6-7], and it is often cited as Cryer's method. It is a successive over-relaxation (SOR) method proposed for the solution of the free-boundary problem for journal bearings; see also [8-9].

Generally, there are some iterative methods for the solution of the LCP, including the projected methods [10-23], the modulus algorithms [24-26] and the modulus-based matrix splitting iterative methods [27-32], see [33] for a survey of the solvers for LCP (1).

In this paper, we study all the fixed point models for linear complementarity problems (LCPs) and present a new model. Finally, we prove the equivalent of the LCP and our fixed-point model.

## 2 The existing fixed-point models

In this section, we study the existing fixed-point model for the LCP.

### 2.1 Projected fixed-point model

Let us consider LCP (1). $LCP(A,q)$ is equivalent to the following zero-finding formulation;

$$\min(z,(Az-q))=0. \tag{2}$$

And the zero-finding formulation is equivalent to the following fixed-point formulation;

$$\max(0,z-(Az-q))=z. \tag{3}$$

This fixed point model used in some iterative methods for LCPs called the projected iterative methods [10-23]. We describe shortly this class of iterative methods.
For any iteration we have the following splitting;

$$A=M-N,$$

then from Eq. (3):

$$\max(0,z^{k+1}-(Mz^{k+1}-Nz^{k}-q))=z^{k+1}.$$

We know that, if $(z^{k+1}-(Mz^{k+1}-Nz^{k}-q))_i<0$ then $z_i^{k+1}=0$, otherwise

$$\begin{aligned}
&(z^{k+1}-(Mz^{k+1}-Nz^{k}-q))_i=z_i^{k+1},\\
&\Rightarrow z_i^{k+1}-(Mz^{k+1}-Nz^{k}-q)_i=z_i^{k+1},\\
&\Rightarrow Mz_i^{k+1}=Nz_i^{k}+(q)_i=((q)_i+(M-A)z_i^{k}),\\
&\Rightarrow Mz_i^{k+1}=(Mz^{k}-(Az^{k}-q))_i.
\end{aligned}$$

Now, for example, if

$$M=D-L,$$

where $D,L$ are diagonal, strictly lower triangular parts of $A$, in order to solve $LCP(A,q)$, we have,

$$(D-L)z_i^{k+1} = ((D-L)z^k - (Az^k - q))_i.$$
$$\Rightarrow Dz_i^{k+1} = (D-L)z_i^k - (Az^k - q)_i + Lz_i^{k+1}.$$
$$\Rightarrow z_i^{k+1} = z_i^k - (D)^{-1}Lz_i^k - (D)^{-1}(Az^k - q)_i + (D)^{-1}Lz_i^{k+1}.$$

Therefore, we get the following formula of the projective iterative method:
$$z^{k+1} = \max(0, z^k - (D)^{-1}((A+L)z^k - Lz^{k+1} - q)).$$

## 2.2 Modulus fixed-point model

Modulus fixed-point model was introduced by Van Bokhoven [24] as follows:
If $I+A$ is nonsingular, then the LCP (1) is equivalent to the fixed point problem of determining $x \in R^n$ satisfying
$$x = f(x) = (I+A)^{-1}(I-A)|x| + (I+A)^{-1}q. \tag{4}$$

More precisely (see the proof of Theorem 9.1 in [1]), if x is a solution of (4), then
$$w = |x| - x, \quad z = |x| + x.$$

Define a solution of (1). On the other hand, if $w, z$ solve (1), then $x = \frac{1}{2}(z - w)$ is a solution of (4). This model was extended by Dong and Jiang [26] to the modified modulus algorithms:
$$\begin{cases} x = f(x) = (\alpha I + A)^{-1}(\alpha I - A)|x| + (\alpha I + A)^{-1}q, \\ \alpha \in R. \end{cases} \tag{5}$$
then,
$$w = \alpha(|x| - x), \quad z = |x| + x.$$

After that, Bai [27] combined the modulus algorithms with the classical iterative methods as follows:
By taking $w = \frac{\Omega}{\gamma}(|x| - x)$, $z = \frac{1}{\gamma}(|x| + x)$ and $A = M - N$, the $LCP(A,q)$ can be equivalently transformed into a system of the following fixed-point equations:
$$x = f(x) = (\Omega + M)^{-1}(Nx + (\Omega - A)|x| + \gamma q). \tag{6}$$

where $\Omega$ is a positive diagonal parameter matrix and $\gamma$ is a positive constant. His work followed by a number of researchers in [28-32].

# 3 A new fixed point model for LCPs

In this section, we will establish a new fixed point model for LCP (1). By definitions of $(x_j)_+ = max\{0, x_j\}$ and $(x_j)_- = \min\{0, x_j\}$ the LCP (1) is equivalent to the fixed point problem of determining $z \in R^n$ satisfying
$$z = f(z) = z + (q - Az)_+ - z^T(q - Az)_-. \tag{7}$$

In the following theorems, we prove an equivalent fixed-point equations (7) and the LCP (1).
**Theorem 1.** Given the fixed-point model (7), and $z \geq 0$. Then $z$ is the solution of LCP (1).

Proof. Let $z \geq 0$. If $(q - Az)_i > 0$, then from Eq.(7) we get:

$$z_i + ((q - Az)_+)_i - z_i \times (0) = z_i .$$

Therefore,

$$z_i + \underbrace{(q - Az)_i}_{>0} = z_i . \qquad (8)$$

We can see that the Eq. (8) has no solution. This means that when $z_i \geq 0$ and $(Az - q)_i < 0$ the LCP (1) is not held.

If $(q - Az)_i < 0$, then from Eq. (7) we obtain:

$$z_i = z_i + \underbrace{((q - Az)_+)_i}_{=0} - (z^T (q - Az)_-)_i$$
$$= z_i - (z^T (q - Az))_i .$$

Since in LCP, $z^T (Az - q) = 0$, then:

$$z_i = z_i - \underbrace{(z^T (q - Az))_i}_{=0} = z_i .$$

If $(q - Az)_i = 0$, then from Eq. (7) we have:

$$z_i + ((q - Az)_+)_i - (z^T (q - Az)_-)_i = z_i + 0 - 0 = z_i$$

Thus, the proof is completed.

**Theorem 2.** If $z^*$ be the solution of LCP (1), then $z^* = f(z^*)$.

Proof. Let $z^*$ is the solution of LCP (1), *i.e*,

$Az^* - q \geq 0,\quad z^* \geq 0,$ and $z^{*T} (Az^* - q) = 0.$

So, when $(Az^* - q)_i > 0,$ then from above LCP we have

$$(z^*)_i = 0. \qquad (9)$$

Also, by Eq. (7):

$$(f(z^*))_i = (z^*)_i + ((q - Az^*)_+)_i - (z^{*T} (q - Az^*)_-)_i$$
$$= (z^*)_i + 0 - (z^{*T} (q - Az^*)_-)_i$$
$$= (z^*)_i + 0 - 0 = (z^*)_i .$$

Then by Eq. (9) we get:

$$(f(z^*))_i = (z^*)_i = 0.$$

Now, when $(Az^* - q)_i = 0,$ then:

$$(f(z^*))_i = (z^*)_i + \underbrace{((q - Az^*)_+)_i}_{=0} - (z^{*T} (q - Az^*)_-)_i$$
$$= (z^*)_i + 0 - (z^{*T} (q - Az^*)_-)_i$$
$$= (z^*)_i + 0 - 0 = (z^*)_i .$$

Thus, the proof is completed.

By two above theorems, we can conclude that the LCP (1) is equivalent to the fixed point model of (7).

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