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Three levels of sustainable production inventory model for defective and deteriorating items with rework under marketplace selling price

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Abstract The intention of the article is to present an imperfect production process inventory system for the decaying items, where the demand rate is depending on advertising cost and price. In this research, an industrial manager produces products in a determined (m) imperfect replenishment cycles and also in each replenishment cycle, the manufacturer manufactures items in three imperfect production cycles and one rework setup, $(M/P_3/R_1)$ inventory system. Rework is one of the important issues in reverse logistics and green supply chain, since it reduces waste, environmental harms and also the overall inventory cost significantly. Here shortages are considered under complete backlogging strategy. The major objective is to establish the optimal advertising cost, optimal cycle time, optimal lot size quantity and optimum selling price with the intend of minimizing the total inventory cost. A numerical example and a sensitivity analysis are recognized to authenticate the theoretical results, and managerial insights for business managers are provided.

Keyword: Three levels of production, imperfect replenishment cycle, defective items, rework, advertising cost

1 Introduction

The inventory system is a major torrent of the operations research which is indispensable in venture and manufacturing. The Economic Production Quantity (EPQ) model is a trouble-free mathematical model to pact with inventory organization problems. In common, roughly all goods are established to be deteriorating with respect to time. On occasion the rate of deterioration is very low, for items such as steel, hardware, glassware and toys, to cause deliberation of deterioration in the determination of optimum lot sizes. Usually, some products have an important rate of decaying, like fish, blood, alcohol, strawberry, gasoline,

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medicine, radioactive chemicals and food grains those deteriorate quickly over time, which could not be overlooked in the decision - making progression of optimal production lot quantity.

In the real - world manufacturing system, the manufacturing of imperfect products is to be projected due to process deterioration, setup errors or other manageable or unmanageable problems (Hinckley, [8]). These imperfect products could be rescheduled to remanufacture. The remanufacture of the imperfect products is the main and important problems in reverse logistics where used products are remanufactured to lessen waste and ecological harms and also the overall inventory costs may be decreased considerably. For illustration, the semiconductor, glass, paper, plastic industries and metal processing employ rework as an unavoidable process in order to attain the necessary quality and quantity levels.

1.1 Literature review on deteriorating items

Deterioration of goods is one of the natural phenomenon’s, probably for manufacturing industries of agricultural products, food items, pharmaceutical items, volatile liquids, etc. Deterioration begins due to evaporation, harm, spoilage, dryness, etc., and it decreases the quality and quantity of stock products. Ghare and Schrader [7] were the primary to integrate the plan of deterioration in inventory control and model. They premeditated an exponentially deteriorating inventory model in terms of constant demand. Later, Perumal and Arivarignan [26] presented an inventory model considering various different rates of manufacturing, deterioration and shortages. Cárdenas Barrón [2] studied a single-step manufacturing inventory model. Later, Bhowmick and Samanta [1] considered a decaying inventory system with two dissimilar rates of production, shortage, and variable replenishment cycle. Sarkar and Sarkar [33] analyzed an inventory system for supply dependent demand and variable decay rate. Sarkar et al. [34] introduced an inventory model beneath variable decay rates. Karthikeyan and Viji [11] proposed an EPQ inventory model for constant decaying items where three dissimilar rates of manufacturing are taken into account. Sarkar and Saren [31] proposed a single warehouse inventory system for variable decay rates. Chowdhury et al. [6] investigated an optimum inventory production policy for decaying products in which the rate of decaying of goods is directly proportional to time. Mashud et al. [19] developed a non-instantaneous inventory system for two different constant rates of deterioration beneath the partial backlogging system. Thangam [46] analyzed an inventory model for delay in payments and imprecise transport in Covid-19 pandemic. Sharma et al. [38] presented the non-instantaneous deterioration model under production policy where the deterioration starts with the beginning time. Karimi and Sadjadi [12] presented a multi-item inventory model for deteriorating items with limited carrier capacity.

1.2 Literature review on defective items

The most basic research work that focused on remanufacturing system was developed by Schrady [36]. After that, the investigators on remanufacturing system have engrossed many research workers. Khouja [15] presented remanufacturing system for economic production lot sizing and deliverance development problem. Koh et al. [16] investigated an economic manufacturing inventory models wherein the trader can block up the order in two different options: orders new items superficially or recovers imperfect items in rework. Jamal et al. [9] introduced two remanufacturing policies. In the initial strategy, imperfect products are

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remanufactured in the same cycle; and in the next strategy, rework is finished behind N replenishment cycles. Yoo et al., [51] considered an EPQ model by means of defective production quality, improper examination and imperfect rework. Taleizadeh et al. [44] investigated an EPQ model by allowing for random defective products, repair failure and service stage constraint. Later, Taleizadeh et al. [45] presented an EPQ inventory model for two joint systems including and excluding rework. Sarkar and Sarkar [32] addressed an EPQ model for decaying products and exponential demand in excess of a finite planning horizon in which the defective products are reworked at a fixed cost to build it as good products. Pal and Adhikari [24] developed an EPQ inventory model for defective quality items in which the rework process is measured but there is no scrap item. Chakrabarty et al. [3] presented an inventory model with production in which the defective item is repaid good quality items. Sekar and Uthayakumar [37] addressed an imperfect producing model allowing for rework and shortages beneath inflation. Xu and Song [50] urbanized integrated optimization of fabrication capacity, raw material ordering and manufacture planning beneath quantity uncertainty. Recently, Öztürk [22] planned an imperfect production process by means of random breakdowns, remanufacturing, and scrutiny costs.

1.3 Literature review on shortages

One unrealistic hypothesis of the traditional EPQ model is that the shortages are not permitted. But in reality, due to manufacture of imperfect items, screening and disposal decaying items and supplementary manufacturing problems, shortages are predictable. Shortages are classified as: absolutely backlogged (Shortage unit is completely fulfilled) and partly backlogged (shortage unit is partly fulfilled). In the recent days, a few customers are eager to stay till the replacement if the waiting time petites whereas others some are impatient and leave somewhere else. To replicate the observable fact, Chang and Dye [4] studied an economic production inventory model considered shortage with limited backlogging in which the rate of backlogging is the reciprocal of a linear function of the waiting time. In recent times, numerous researchers have focused on partly/absolutely backlogging shortages. Wee et al. [48] presented an integrated inventory model for decay products wherein shortages are absolutely backlogged. Roy et al. [27] analyzed an EPQ model for imperfect products including partly backlogging shortages. An inventory model allowing for time dependent demand and time unreliable holding cost beneath partly backlogging shortage was addressed by Mishra et al. [21]. Pal et al. [25] analyzed a model interrelated to the inventory system of Sarkar et al. [35] introducing a multi-manufacturing setup for decaying products, ramp type demand and inflation when shortages occur in the store beneath the finite time planning horizon. Taleizadeh et al. [43] investigated a four different and novel sustainable EPQ models which, considering different kinds of shortage circumstances. Khara et al [14] formulated an imperfect production model considering advanced payment and trade credit facilities. Based on the research work stated above, it could be decided that a substantial amount of research works have been urbanized for multi-production cycles taking remanufacturing and fully backlogging shortage units.

1.4 Novelty, contribution and objective of the models

The innovation behind formulating this EPQ inventory system is: (a) A three different rate of production inventory model for decaying products; (b) the rate of production is limited and

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proportional to the demand factor; (c) Demand depends on both advertising cost and selling price; (d) Shortage is permitted under backlogging completely .

The major involvement of this inventory model is to develop an advertising cost and price-dependent demand EPQ model while shortage is permitted. The replenishment begins at one rate, and subsequent to a time it may be relocated to a different rate of production, then again it may be transferred to a third rate of production. This type of circumstances is desirable in the sense that by starting at a low rate of production, a huge stock of industrial unit made products at the first stage is avoided, leading to a reduce in the cost of carrying. The major intention is to establish the optimum cost of advertising, optimum time for replenishment cycle, and optimal selling price under the aim of minimizing the total inventory cost .

This article is constructed as follows: Section 2 establishes the assumptions and notations used in the inventory system. Section 3 depicts that the problem description and the mathematical formation of this inventory system. The developed inventory model is further demonstrated through one numerical example and the proof of convexity is graphically offered in Section 4. Section 5 gives a sensitivity analysis, including managerial insights. Finally, the conclusions of this model and future research guideline are presented in the Section 6 .

The contributions of the different researchers from the existing literature are provided in Table 1 .

Table 1 Summary of interrelated literature for multi-production-run, defective production, rework and shortage.

Author(s)	Multi-replenishment cycle	Defective production	Rework	Three production setups	Shortage
Chiu et al. (2007)	√	√	√		
Sana et al. (2007)			√		
Sarkar et al. (2010)		√	√		
Widyadana &Wee (2012)	√	√	√		
Krishnamoorthi & Panayappan (2012)		√	√		
Sarkar (2012)		√			
Taleizadeh et al. (2013a)		√	√		√
Tai (2013)		√	√		√
Singh et al. (2014)	√	√	√		
Sarkar et al. (2014b)		√	√		√
Pal et al. (2014)	√				
Taleizadeh et al. (2014)		√	√		√
Pal et al. (2015)	√				√
Li et al. (2015)		√	√		√
Jawla and Singh (2016)	√	√	√		
Viji and Karthikeyan (2016)				√	√
Khanna et al. (2017)		√	√		
Mishra (2018)				√	√
Present model	√	√	√	√	√

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2 Notation and assumptions

2.1 Notation

Decision variables

A: Cost of frequency advertisement per cycle.
T_7: Model cycle time.
s: Market selling price per unit.

Parameters

Q_1 : Production quantity (units) at time T_1.
Q_2 : Production quantity (units) at time T_2.
Q_3 : Production quantity (units) at time T_3.
Q : Production quantity (units) at time T_6.
S : Backordering units at time T_5.
S^* : Optimum backordering quantity (units) at time T_5.
Q^* : Optimum production lot size (units) at timeT_6.
$D(A,p)$: Rate of demand (units/time unit).
k_p : Production cost (\$/unit/time unit).
k_s : Setup cost (\$/cycle/run).
k_a : Cost for every advertisement.
h_s : Cost for holding serviceable products (\$/unit/time unit).
h_r : Cost for holding defective items (\$/unit/time unit).
C_s : Cost for shortage units (\$/unit/time unit).
T_1, T_2, T_3, T_4, and T_5 : Cycle time.
T_1^*, T_2^*, T_3, T_4^*, and T_5^* : Optimum inventory cycle time (time units).
Q_1^*, Q_2^*, and Q_3^* : Optimum production lot size (units).
$TCF(A,T_7,s)$: Total average inventory cost per production cycle.

2.2 Assumptions

To develop the proposed inventory model, we apply the following assumptions:

1.The production rate is $P = \lambda D(A,s)$, where $\lambda(> 1)$ is a constant. Therefore, $P > D(A,s)$. The production rate for each replenishment cycle is given by

$$P = \begin{cases} p_1 & when\ t \in [0,T_1] \\ p_2 & when\ t \in [T_1,T_2] \\ p_3 & when\ t \in [T_2,T_3] \\ p_4 & when\ t \in [T_3,T_4] \end{cases}$$

where $p_1 = (\alpha\lambda - 1)D(A,s) - \theta$, $p_2 = a(\alpha\lambda - 1)D(A,s) - \theta$, $p_3 = b(\alpha\lambda - 1)D(A,s) - \theta$ $p_4 = p_r - D(A,s) - \theta$ and α is a portion of good quality items.

2.The rate of demand is a function of selling parameters, with the incidence of advertising A, being a positive integer, and selling price (p), i.e., $D(A,s) = (r - ls)A^{\eta}$, where $r(> 0)$ is a scaling factor, l is the index of price elasticity, and η $(0 \leq \eta << 1)$ is the shape parameter.

3.Single product deterministic three different rates of producing inventory system with demand rate $D(A,s)$.

4.The rate of deterioration per unit time is a constant fraction θ $(0 < \theta << 1)$

5.Deterioration of goods begins as soon as it arrives into the inventory.

6.Shortage is permitted and backlogged completely in all replenishment cycles.
7.Model is developed under finite time planning horizon.
8.The rate of production of serviceable items and the rate rework must be greater than the demand rate.
9. Machine's breakdown does not occur during replenishment as well as rework run time.
10. All the imperfect products and deteriorated products screened out are disposed.

3 Problem definition and mathematical formulation of this inventory model

During $[0, T_1]$ the production rate is p_1 and the demand rate is $D(A,\ s)$. When time t where $t \in [T_1,\ T_2]$, the inventory put up at the production rate p_2. When time t where $t \in [T_2,\ T_3]$, the inventory builds up at the rate of production p_3. When time T_3, the natural production ends, the rework process begins. When $t = T_4$, where $t \in [T_3,\ T_4]$, the level of inventory reaches the level of maximum (I_{ms}). At time $[T_4,\ T_5]$, the level of inventory diminishes because of the demand and deterioration of items, during $[T_5,\ T_6]$ the inventory system undergoes shortages and during $[T_6,\ T_7]$, the production-run resumes to satisfy the shortage units. The behavior of the inventory system is drawn in Figure 1 and 2.
In truth, when the production machines went through a long-run production, the various types of issues arise that results the manufacturing of imperfect products. These imperfect products are remanufactured between time $t = 0$ to $t = T_4$ during the time interval $(0,\ T_4)$. When time $t = T_4$, the level of inventory attains the level of maximum (I_{ms}). During $(0,\ T_5)$, the level of inventory reduces owing to the rate of demand and deterioration, during $(0,\ T_6)$ the model meets shortages and during $(0,\ T_7)$ production-run resumes to satisfy the shortage units. We inspect a typical producer-consumer inventory system where the production process of manufacturer is imperfect and examination on deteriorated items is perfect. The company produces p (rate of production) units and concurrently inspects that the αp items are serviceable and the $(1-\alpha)p$ items are defective per unit time during a production-run time (Widyadana &Wee [49]). The serviceable products are stocked in 'serviceable inventory'. At the same time, the defective products are stocked in 'recoverable inventory' until the regular production-run ends. The rework process begins immediately after the determined replenishment-run ends. Owing to inappropriate inventory circumstance or other causes, both serviceable and recoverable inventories undergo deterioration. In the two inventories, inspection is carried to discover and dispose the deteriorating products. After the serviceable inventory level comes to vanish, shortages are taking place and which are backlogged completely in each production cycle. The same m production cycles are considered in a finite time planning horizon (H). The behaviors of the level of inventory of serviceable and defective products for three different production-run and one rework process are demonstrated in Fig. 1and Fig. 2 respectively.

3.1. Inventory level of good quality items

The pictorial representation of the inventory level of good quality products when time T_i (i= 1, 2, …, 7) in a replenishment cycle is drawn in Fig-1. The differential equations of the level of inventory of serviceable products are obtained as given below:

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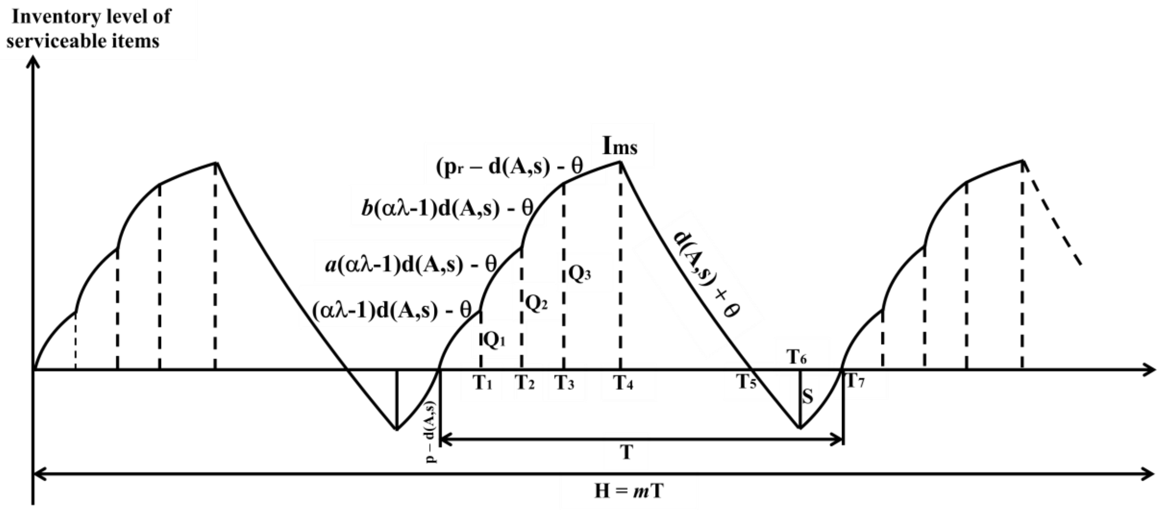

Fig. 1 Serviceable item's inventory level for m = 3.

The level of inventory of serviceable products in a production-run during $[0, T_1]$ can be formulated as:

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = (\alpha\lambda - 1)D(A, s) \qquad 0 \le t \le T_1 \tag{1}$$

where $\theta I_1(t)$ is a fraction of deteriorated products discovered from the serviceable inventory at time t where $0 \le t \le T_1$.

The level of serviceable inventory in a production-run during $[0, T_2]$ is represented by the following equation:

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = a(\alpha\lambda - 1)D(A, s) \qquad T_1 \le t \le T_2 \tag{2}$$

The level of serviceable inventory in a production-run during $[0, T_3]$ can be derived as:

$$\frac{dI_3(t)}{dt} + \theta I_3(t) = b(\alpha\lambda - 1)D(A, s) \qquad T_2 \le t \le T_3 \tag{3}$$

The level of serviceable inventory in a rework process period $[0, T_4]$ can be formed as:

$$\frac{dI_4(t)}{dt} + \theta I_4(t) = p_r - D(A, s) \qquad T_3 \le t \le T_4 \tag{4}$$

The level of serviceable inventory in a non-production-run during $[0, T_5]$ can be obtained by the following equation:

$$\frac{dI_5(t)}{dt} + \theta I_5(t) = -D(A, s) \qquad T_4 \le t \le T_5 \tag{5}$$

The level of good quality items inventory during shortage accumulate time $[0, T_6]$ is stated by the following differential equation:

$$\frac{dI_6(t)}{dt} = -D(A, s) \qquad T_5 \le t \le T_6 \tag{6}$$

The level of serviceable inventory during shortage down time $[0, T_7]$ is given by the following equation:

$$\frac{dI_7(t)}{dt} = (\lambda - 1)D(A, s) \qquad T_6 \le t \le T_7 \tag{7}$$

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Boundary conditions:
$I_1(0) = I_5(T_5) = I_6(0) = I_7(T_7) = 0, I_1(T_1) = I_2(0) = Q_1, I_2(T_2) = I_3(0) = Q_2,$
$I_3(T_3) = I_4(0) = Q_3, I_4(T_4) = I_5(0) = I_{ms}.$ (8)
Solving the differential equations (1) – (7) yields

$$I_1(t) = \frac{(\alpha\lambda-1)D(A,s)}{\theta}\left[1-e^{-\theta t}\right] \qquad 0 \le t \le T_1 \tag{9}$$

$$I_2(t) = \frac{a(\alpha\lambda-1)D(A,s)}{\theta}\left[1-e^{-\theta t}\right] \qquad T_1 \le t \le T_2 \tag{10}$$

$$I_3(t) = \frac{b(\alpha\lambda-1)D(A,s)}{\theta}\left[1-e^{-\theta t}\right] \qquad T_2 \le t \le T_3 \tag{11}$$

$$I_4(t) = \frac{p_r - D(A,s)}{\theta}\left[1-e^{-\theta t}\right] \qquad T_3 \le t \le T_4 \tag{12}$$

$$I_5(t) = \frac{D(A,s)}{\theta}\left[e^{\theta(T_5-t)}-1\right] \qquad T_4 \le t \le T_5 \tag{13}$$

$$I_6(t) = -D(A,s)[T_5 - t] \qquad T_5 \le t \le T_6 \tag{14}$$

$$I_7(t) = (\lambda-1)D(A,s)[T_7 - t] \qquad T_6 \le t \le T_7 \tag{15}$$

Maximum production lot size quantity (Q_1):
During the production-run time $[0, T_1]$, the maximum economic production lot size (Q_1) is calculated with the help of the boundary condition $I_1(T_1) = Q_1$ in the equation (9).

$$I_1(T_1) = Q_1 = \frac{(\alpha\lambda-1)D(A,s)}{\theta}\left[1-e^{-\theta T_1}\right] \tag{16}$$

By using the exponential function and ignoring the higher powers of θ $(0 < \theta << 1)$, we get
$$Q_1 = (\alpha\lambda-1)D(A,s)T_1 \tag{17}$$
Maximum production lot size quantity (Q_2):
During the production-run time $[0, T_2]$, the maximum replenishment lot size quantity (Q_2) is derived with the help of the boundary condition $I_2(T_2) = Q_2$ in the equation (10).
$$I_2(T_2) = Q_2 = \frac{a(\alpha\lambda-1)D(A,s)}{\theta}\left[1-e^{-\theta T_2}\right] \tag{18}$$
By using the exponential function and ignoring the higher powers of θ $(0 < \theta << 1)$, therefore
$$Q_2 = a(\alpha\lambda-1)D(A,s)T_2 \tag{19}$$
Maximum production lot size quantity (Q_3):
During the production-run time $[0, T_3]$, the maximum replenishment lot size quantity (Q_3) is gotten by using the boundary condition $I_3(T_3) = Q_3$ in the equation (11).
$$I_3(T_3) = Q_3 = \frac{b(\alpha\lambda-1)D(A,s)}{\theta}\left[1-e^{-\theta T_3}\right] \tag{20}$$
By using the exponential function and ignoring the higher powers of θ $(0 < \theta << 1)$, therefore
$$Q_3 = b(\alpha\lambda-1)D(A,s)T_3 \tag{21}$$
Maximum serviceable inventory (I_{ms}):
During the replenishment-run time $[0, T_4]$, the maximum serviceable inventory (I_{ms}) is obtained by using the condition $I_4(T_4) = I_{ms}$ in the equation (12).

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$$I_4(T_4) = I_{ms} = (p_r - D(A,s))T_4 \tag{22}$$

Maximum shortage level (S):

During the shortage accumulate time $[0, T_6]$, the maximum level of shortage is calculated by using the condition $I_6(T_6) = -S$ in the equation (14).

$$I_6(T_6) = -S \Rightarrow D(A,s)[T_6 - T_5] = -S \tag{23}$$

From equation (15), we obtain

$$I_7(T_6) = -S \Rightarrow (\lambda - 1)D(A,s)[T_7 - T_6] = -S$$

Therefore, $T_6 = \frac{T_5 + (\lambda-1)T_7}{\lambda}$ (24)

3.2 Inventory level of defective products

The pictorial representation of the inventory level of the imperfect products is drawn in the Fig. 2. The governing differential equations of the inventory level of imperfect items are:

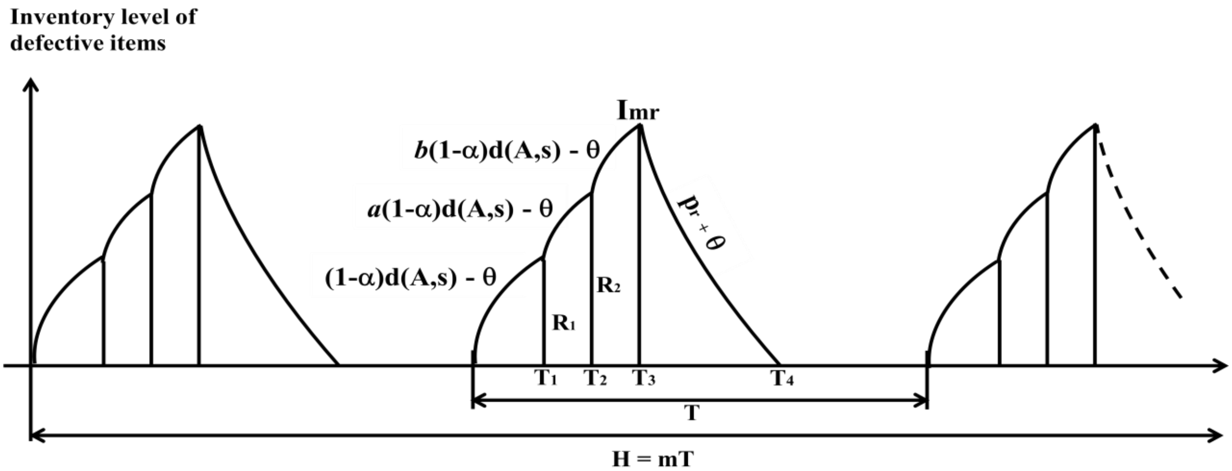

Fig. 2 Deteriorative inventory level of defective products for m = 3.

The level of inventory of imperfect products in a production-run time $(0, T_1)$ can be formulated as:

$$\frac{dI_{r1}(t)}{dt} + \theta I_{r1}(t) = (1-\alpha)\lambda D(A,s) \qquad 0 \le t \le T_1 \tag{25}$$

where $\theta I_{r1}(t)$ is a fraction of deteriorated items discovered from the recoverable inventory at time t where $0 \le t \le T_1$.

The level of inventory of imperfect products in a production-run time $(0, T_2)$ is represented by the equation:

$$\frac{dI_{r2}(t)}{dt} + \theta I_{r2}(t) = a(1-\alpha)\lambda D(A,s) \qquad T_1 \le t \le T_2 \tag{26}$$

The level of inventory of imperfect products in a production-run time $(0, T_3)$ can be modeled as:

$$\frac{dI_{r3}(t)}{dt} + \theta I_{r3}(t) = b(1-\alpha)\lambda D(A,s) \qquad T_2 \le t \le T_3 \tag{27}$$

The level of inventory of imperfect products in a rework process period $(0, T_4)$ can be formed as:

$$\frac{dI_{r4}(t)}{dt} + \theta I_{r4}(t) = -p_r \qquad T_3 \le t \le T_4 \tag{28}$$

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Boundary conditions:

$I_{r1}(T_1) = I_{r2}(0) = R_1,\ I_{r2}(T_2) = I_{r3}(0) = R_2, I_{r3}(T_3) = I_{r4}(0) = I_{mr},\ I_{r4}(T_4) = 0$ (29)

The solution of the differential equations (21) – (24) are given below:

$$I_{r1}(t) = \frac{(1-\alpha)\lambda D(A,s)}{\theta}\left[1 - e^{-\theta t}\right] \qquad 0 \le t \le T_1 \qquad (30)$$

$$I_{r2}(t) = \frac{a(1-\alpha)\lambda D(A,s)}{\theta}\left[1 - e^{-\theta t}\right] \qquad T_1 \le t \le T_2 \qquad (31) \qquad (28)$$

$$I_{r3}(t) = \frac{b(1-\alpha)\lambda D(A,s)}{\theta}\left[1 - e^{-\theta t}\right] \qquad T_2 \le t \le T_3 \qquad (32)$$

$$I_{r4}(t) = \frac{p_r}{\theta}\left[e^{\theta(T_4-t)} - 1\right] \qquad T_3 \le t \le T_4 \qquad (33)$$

Maximum defective items inventory (R_1):

During the production-run interval $(0, T_1)$, the maximum inventory level of imperfect items (R_1) is calculated with the help of the boundary condition $I_{r1}(T_1) = R_1$ in the equation (27) which is given by:

$$R_1 = \frac{(1-\alpha)\lambda D(A,s)}{\theta}\left[1 - e^{-\theta T_1}\right] \qquad (34)$$

By using the exponential function and ignoring the higher powers of θ $(0 < \theta << 1)$, we get

$$R_1 = (1-\alpha)\lambda D(A,s)T_1 \qquad (35)$$

Maximum defective items inventory (R_2):

During the production-run time interval $(0, T_2)$, the maximum inventory level of imperfect products (R_2) can be obtained by using the boundary condition $I_{r2}(T_2) = R_2$ in the equation (28) which is given by:

$$R_2 = \frac{a(1-\alpha)\lambda D(A,s)}{\theta}\left[1 - e^{-\theta T_2}\right] \qquad (36)$$

By using the exponential function and ignoring the higher powers of θ (0 < θ << 1), we get

$$R_2 = a(1-\alpha)\lambda D(A,s)T_2 \qquad (37)$$

Maximum defective items inventory (I_{mr}):

During the replenishment-run time interval $(0, T_3)$, the maximum inventory level of imperfect items (I_{mr}) is calculated by using the condition $I_{r3}(T_3) = I_{mr}$ in the equation (32) which is given by:

$$I_{mr} = \frac{b(1-\alpha)\lambda D(A,s)}{\theta}\left[1 - e^{-\theta T_3}\right]$$

By using the exponential function and ignoring the higher powers of θ (0 < θ << 1), we get

$$I_{mr} = b(1-\alpha)\lambda D(A,s)T_3 \qquad (38)$$

The rework-run time (T_4):

Since $I_{r3}(T_3) = I_{r4}(T_3)$, we obtain

$$\frac{b(1-\alpha)\lambda D(A,s)}{\theta}\left[1 - e^{-\theta T_3}\right] = \frac{p_r}{\theta}\left[e^{\theta(T_4-T_3)} - 1\right]$$

By using the exponential function and ignoring the higher powers of θ (0 < θ << 1), the above equation reduced as:

$$T_4 = \frac{(p_r + b(1-\alpha)\lambda D(A,s))T_3}{p_r} \qquad (39)$$

(i) *The production setup cost (PSC)*

At the starting of the production process, the supplier has to get the equipment ready. The first production setup occurs when $t = 0$ in the interval $(0, T_1)$.

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$PSC = k_s$ (40)

(ii) *The rework setup cost (RSC):*

The setup cost for rework process takes place at the beginning of the rework-run time, i.e., when $t = 0$ in the interval $(0, T_4)$.

RSC = k_r (41)

(iii) *The production cost (CP):*

CP = $k_p D(A,s)$ (42)

(iv) *The advertising cost (ADC):*

$ADC = \; k_a A$ (43)

(v) *The holding cost of serviceable items (HCs):*

Since it is essential to maintain the serviceable products in stock during the periods $(0, T_1)$, $(0, T_2)$, $(0, T_3)$, $(0, T_4)$ and $(0, T_5)$, the holding cost is calculated in the periods. During $(0, T_6)$, the items meet the shortage so there is no product to stock and during $(0, T_7)$, the products are produced which are used to overcome the backorder so no products are required to be held or stored during this time period.

$$\text{HCs} \;= h_s \int_0^{T_1} I_1(t)dt + h_s \int_{T_1}^{T_2} I_2(t)dt + h_s \int_{T_2}^{T_3} I_3(t)dt + h_s \int_{T_3}^{T_4} I_4(t)dt + \int_{T_4}^{T_5} I_5(t)dt$$

$$\text{HCs} \;= h_s \begin{bmatrix} \int_0^{T_1} \frac{(\alpha\lambda-1)D(A,s)}{\theta}\left[1-e^{-\theta t}\right]dt + \int_{T_1}^{T_2} \frac{a(\alpha\lambda-1)D(A,s)}{\theta}\left[1-e^{-\theta t}\right]dt \\ +\int_{T_2}^{T_3} \frac{b(\alpha\lambda-1)D(A,s)}{\theta}\left[1-e^{-\theta t}\right]dt + \int_{T_3}^{T_4} \frac{p_r - D(A,s)}{\theta}\left[1-e^{-\theta t}\right]dt \\ +\int_{T_4}^{T_5} \frac{D(A,s)}{\theta}\left[e^{\theta(T_5-t)}-1\right]dt \end{bmatrix}$$

Ignoring the terms involving the second and higher powers of θ as $0 < \theta << 1$, we get

$$\text{HCs} = h_s \begin{bmatrix} (\alpha\lambda-1)D(A,s)T_1^2 + a(\alpha\lambda-1)D(A,s)(T_2^2-T_1^2) + b(\alpha\lambda-1)D(A,s)(T_3^2-T_2^2) \\ +(p_r \;-\; D(A,s))(T_4^2-T_3^2) + D(A,s)(T_5^2-T_4^2) \end{bmatrix}$$

$$\text{HCs} \;= h_s \begin{bmatrix} (\alpha\lambda-1)D(A,s)((1-a)T_1^2 + (a-b)T_2^2 + bT_3^2) \\ +(p_r \;-\; D(A,s))(T_4^2-T_3^2) + D(A,s)(T_5^2-T_4^2) \end{bmatrix} \quad (44)$$

(vi) *The holding cost of defective items (HC_d):*

Since it is essential to maintain the defective products in stock during the time periods $(0, T_1)$, $(0, T_2)$, $(0, T_3)$, $(0, T_4)$ and $(0, T_7)$, the holding cost is planned during these periods. Hence, the holding cost of defective products during the above periods is given by:

$$HC_d = h_r \int_0^{T_1} I_{r1}(t)dt + h_r \int_{T_1}^{T_2} I_{r2}(t)dt + h_r \int_{T_2}^{T_3} I_{r3}(t)dt + h_r \int_{T_3}^{T_4} I_{r4}(t)dt$$

$$HC_d = h_r \begin{bmatrix} \int_0^{T_1} \frac{(1-\alpha)\lambda D(A,s)}{\theta}\left[1-e^{-\theta t}\right]dt + \int_{T_1}^{T_2} \frac{a(1-\alpha)\lambda D(A,s)}{\theta}\left[1-e^{-\theta t}\right]dt \\ +\int_{T_2}^{T_3} \frac{b(1-\alpha)\lambda D(A,s)}{\theta}\left[1-e^{-\theta t}\right]dt + \int_{T_3}^{T_4} \frac{p_r}{\theta}\left[e^{\theta(T_4-t)}-1\right]dt \end{bmatrix}$$

$$HC_d = h_r \begin{bmatrix} (1-\alpha)\lambda D(A,s)T_1^2 + a(1-\alpha)\lambda D(A,s)(T_2^2-T_1^2) \\ +b(1-\alpha)\lambda D(A,s)(T_3^2-T_2^2) + \frac{p_r}{\theta}T_4^2 \end{bmatrix}$$

$$HC_d = h_r \left[(1-\alpha)\lambda D(A,s)[(1-a)T_1^2 + (a-b)T_2^2 + bT_3^2] + \frac{p_r}{\theta}T_4^2\right] \quad (45)$$

(vii) *The cost of deterioration (DC):*

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The products are stocked in an inventory during $(0, T_1)$, $(0, T_2)$, $(0, T_3)$, $(0, T_4)$ and $(0, T_5)$ to meet consumer's demand. The products are deteriorating during the storage time periods. Hence, the deterioration cost for the above time periods is presented by:

$$\text{DC} = \theta D_c \left[\int_0^{T_1} I_1(t)dt + h_s \int_{T_1}^{T_2} I_2(t)dt + h_s \int_{T_2}^{T_3} I_3(t)dt + h_s \int_{T_3}^{T_4} I_4(t)dt + \int_{T_4}^{T_5} I_5(t)dt \right]$$

$$\text{DC} = \theta D_c \begin{bmatrix} \int_0^{T_1} \frac{(\alpha\lambda-1)D(A,s)}{\theta}\left[1-e^{-\theta t}\right] dt + \int_{T_1}^{T_2} \frac{a(\alpha\lambda-1)D(A,s)}{\theta}\left[1-e^{-\theta t}\right]dt \\ + \int_{T_2}^{T_3} \frac{b(\alpha\lambda-1)D(A,s)}{\theta}\left[1-e^{-\theta t}\right]dt + \int_{T_3}^{T_4} \frac{p_r - D(A,s)}{\theta}\left[1-e^{-\theta t}\right]dt \\ + \int_{T_4}^{T_5} \frac{D(A,s)}{\theta}\left[e^{\theta(T_5-t)}-1\right]dt \end{bmatrix}$$

$$\text{DC} = \theta D_c \begin{bmatrix} (\alpha\lambda-1)D(A,s)((1-a)T_1^2 + (a-b)T_2^2 + bT_3^2) \\ +(p_r - D(A,s))(T_4^2 - T_3^2) + D(A,s)(T_5^2 - T_4^2) \end{bmatrix} \quad (46)$$

(viii) *The shortage cost (CS):*

In this proposed model, shortages occur during the time interval $(0, T_6)$ and to overcome the backorders owing to shortages items are manufactured during $(0, T_7)$. Hence, the total time periods for shortages are $(0, T_6)$ and $(0, T_7)$.

$$\text{CS} = c_s \int_{T_5}^{T_6} I_6(t)dt + c_s \int_{T_6}^{T_7} I_7(t)dt$$

$$\text{CS} = c_s \int_{T_5}^{T_6} D(A,s)[t-T_5]dt + c_s \int_{T_6}^{T_7} (\lambda-1)D(A,s)[T_7-t]\,dt$$

$$\text{CS} = c_s\, D(A,s) \left[\left(\frac{t^2}{2} - T_5 t\right)_{T_5}^{T_6} + (\lambda-1)\left(T_7 t - \frac{t^2}{2}\right)_{T_6}^{T_7} \right]$$

$$\text{CS} = \frac{c_s\, D(A,s)}{2}\left[(T_6-T_5)^2 + (\lambda-1)(T_7-T_6)^2\right] \quad (47)$$

Total cost function($\mathbf{TC_F}$):

Total cost function = Setup cost of production + Setup cost of rework + Cost of advertising + Cost of production + Cost of holding of serviceable items + Cost of holding of defective items + Cost of deterioration + Cost of shortage.

$TC_F(A, T_1, T_2, T_3, T_4, T_5, T_7, s) = \left[mCP + (1/T_7)[PSC + RSC + m(ADC + HC_s + HC_d + DC + CS)]\right]$

$TC_F(A, T_1, T_2, T_3, T_4, T_5, T_6, T_7, s) =$

$$\begin{bmatrix} mk_p D(A,s) + \dfrac{k_r + k_s}{T_7} + \dfrac{mk_a A}{T_7} + \dfrac{c_s\, D(A,s)}{2}\left[(T_6-T_5)^2 + (\lambda-1)(T_7-T_6)^2\right] \\ \dfrac{m(h_s + \theta D_c)}{T_7}\begin{bmatrix} (\alpha\lambda-1)D(A,s)((1-a)T_1^2 + (a-b)T_2^2 + bT_3^2) \\ +(p_r - D(A,s))(T_4^2 - T_3^2) + D(A,s)(T_5^2 - T_4^2) \end{bmatrix} \\ \dfrac{mh_r}{T_7}\left[(1-\alpha)\lambda D(A,s)[(1-a)T_1^2 + (a-b)T_2^2 + bT_3^2] + \dfrac{p_r}{\theta}T_4^2\right] \end{bmatrix}$$

Let us assume that $T_1 = \mathcal{u}T_5$, $T_2 = \mathcal{v}T_5$ and $T_3 = \mathcal{w}T_5$.

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$$TC_F(A,T_5,T_7,p) = \left[\begin{array}{c} k_pD(A,s)+(k_r+k_s)\left(\frac{1}{T_7}\right)+\ mk_a\ \left(\frac{A}{T_7}\right)+D(A,s)(T_7-T_5) \\ \left\{\frac{mc_s\ (\lambda-1)}{2\lambda}+m(\alpha\lambda-1)\left[(1-a)u^2+(a-b)v^2+bw^2\right]+m(h_s+\theta D_c)(2\lambda w^2b(1-\alpha)-w^2+1)\right. \\ \left.+m\lambda h_r(1-\alpha)[u^2+a(v^2-u^2)+b(w^2-v^2)]\right\}\left(\frac{D(A,s)T_5^2}{T_7}\right) \\ +\frac{m(h_s+\theta D_c)w^2b\lambda(1-\alpha)}{p_r^2}(b(1-\alpha)\lambda-4p_r)\left(\frac{D(A,s)^2T_5^2}{T_7}\right)-\frac{m(h_s+\theta D_c)w^2}{p_r^2}b^2\lambda^2(1-\alpha)^2\left(\frac{D(A,s)^3T_5^2}{T_7}\right)\end{array}\right]$$

$$TC_F(A,T_5,T_7,p) = \left[\begin{array}{c} k_p(r\ -\ lp)A^\eta+(k_r+k_s)\left(\frac{1}{T_7}\right)+\ mk_a\ \left(\frac{A}{T_7}\right)+C_1\left(\frac{(r\ -lp)A^\eta T_5^2}{T_7}\right) \\ +C_2(r\ -\ lp)A^\eta(T_7-T_5)\ +\ C_3(r\ -\ lp)^2A^{2\eta}\left(\frac{T_5^2}{T_7}\right)-C_4(r\ -\ lp)^3A^{3\eta}\left(\frac{T_5^2}{T_7}\right)\end{array}\right] \tag{48}$$

$$C_1 = \left[\begin{array}{c}\frac{mc_s\ (\lambda-1)}{2\lambda}+m(\alpha\lambda-1)[(1-a)u^2+(a-b)v^2+bw^2] \\ +m(h_s+\theta D_c)(2\lambda w^2b(1-\alpha)-w^2+1) \\ +m\lambda h_r(1-\alpha)[u^2+a(v^2-u^2)+b(w^2-v^2)]\end{array}\right]$$

$$C_2=\frac{mc_s\ (\lambda-1)}{\lambda}$$

$$C_3=\frac{m(h_s+\theta D_c)w^2b\lambda(1-\alpha)}{p_r^2}\left(b(1-\alpha)\lambda-4p_r\right)$$

$$C_4=\frac{m(h_s+\theta D_c)w^2}{p_r^2}b^2\lambda^2(1-\alpha)^2$$

$$D(A,s)\ =\ (r\ -\ ls)A^\eta$$

Solution procedure

For obtaining the optimal solution of this EPQ model, we establish and prove the following theorems:

Theorem 3.1

When inventory cycle time T_5, inventory cycle time T_7, and market selling price s are fixed, the total cost function $TC_F(A,T_5,T_7,s)$ is convex with respect to A.

Proof: The partial derivatives of the first and second order of the total inventory cost function $TC_F(A,T_5,T_7,s)$ with respect to A, are given below:

$$\frac{\partial TC_F(A,T_5,T_7,s)}{\partial A} = \left[\begin{array}{c} k_p(r\ -\ ls)\eta A^{\eta-1}+C_1\left(\frac{(r\ -\ ls)\eta A^{\eta-1}T_5^2}{T_7}\right)+\ C_2(r\ -\ ls)\eta A^{\eta-1}(T_7-T_5) \\ +mk_a\ \left(\frac{1}{T_7}\right)+\ 2C_3(r\ -\ ls)^2\eta A^{2\eta-1}\left(\frac{T_5^2}{T_7}\right)-3C_4(r\ -\ ls)^3\eta A^{3\eta-1}\left(\frac{T_5^2}{T_7}\right)\end{array}\right]$$

The value of A is obtained from the equation $\frac{\partial TC_F(A,T_5,T_7,s)}{\partial A}=0$. The optimum value exists if $\frac{\partial^2 TC_F(A,T_5,T_7,s)}{\partial A^2}>0$. That is.,

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$$\frac{\partial^2 TC_F(A,T_5,T_7,s)}{\partial A^2} = \begin{bmatrix} k_p(r - ls)\eta(\eta-1)A^{\eta-2} + C_1\left(\frac{(r - lp)\eta(\eta-1)A^{\eta-2}T_5^2}{T_7}\right) \\ + C_2(r - ls)\eta(\eta-1)A^{\eta-2}(T_7 - T_5) + 2C_3(r - lp)^2\eta(2\eta-1)A^{2\eta-2}\left(\frac{T_5^2}{T_7}\right) \\ -3C_4(r - ls)^3\eta(3\eta-1)A^{3\eta-2}\left(\frac{T_5^2}{T_7}\right) \end{bmatrix} > 0$$

Since the frequency of advertising cost A is a positive integer, $0 < \eta < 1$ and according to above second derivative equation, the total inventory cost function is convex with respect to A.

Theorem 3.2

When A, T_6, and s are fixed, the total inventory cost per unit time $TC_F(A,T_5,T_7,s)$ is convex with respect to T_5.

Proof: The partial derivatives of the first and second order of the total inventory cost function $TC_F(A,T_5,T_7,s)$ with respect to T_5 are given below:

$$\frac{\partial TC_F(A,T_5,T_7,s)}{\partial T_5} = \begin{bmatrix} 2C_1\left(\frac{(r-ls)A^\eta T_5}{T_7}\right) + C_2(r - ls)A^\eta T_7 \\ + 2C_3(r - ls)^2A^{2\eta}\left(\frac{T_5}{T_7}\right) - 2C_4(r - ls)^3A^{3\eta}\left(\frac{T_5}{T_7}\right) \end{bmatrix}$$

$$\frac{\partial^2 TC_F(A,T_5,T_7,s)}{\partial {T_5}^2} = 2C_1\left(\frac{(r-ls)A^\eta}{T_7}\right) + 2C_3(r - ls)^2A^{2\eta}\left(\frac{1}{T_7}\right) - 2C_4(r - ls)^3A^{3\eta}\left(\frac{1}{T_7}\right) > 0$$

This fulfills the proof of Theorem 3.2.

The value of T_5 is obtained from the equation $\frac{\partial TC_F(A,T_5,T_7,s)}{\partial T_5} = 0$. That is.,

$$2C_1\left(\frac{(r-ls)A^\eta T_5}{T_7}\right) + C_2(r - ls)A^\eta T_7 + 2C_3\left(r - ls\right)^2A^{2\eta}\left(\frac{T_5}{T_7}\right) - 2C_4\left(r - ls\right)^3A^{3\eta}\left(\frac{T_5}{T_7}\right) = 0$$

This implies that $T_5 = C_5T_7^2$ where $C_5 = \frac{C_2}{2[C_4(r-ls)^2A^{2\eta} - C_1 - C_3(r-ls)A^\eta]}$

Putting $\mathrm{T}_5 = \mathrm{C}_5\mathrm{T}_7^2$ in the total inventory cost function $\mathrm{TC_F}(\mathrm{A},\mathrm{T}_5,\mathrm{T}_7,\mathrm{s})$, it becomes

$$TC_F(A,T_7,s) = \begin{bmatrix} k_p(r - ls)A^\eta + (k_r + k_s)\left(\frac{1}{T_7}\right) + mk_a\left(\frac{A}{T_7}\right) + C_1(r - ls)A^\eta C_5^2\,T_7^3 \\ + C_2(r - ls)A^\eta(T_7 - C_5T_7^2) + C_3(r - ls)^2A^{2\eta}C_5^2T_7^3 - C_4(r - ls)^3A^{3\eta}C_5^2T_7^3 \end{bmatrix}$$

Theorem 3.3

When A and price s are fixed, the total inventory cost per unit time $TC_F(A,T_7,s)$, is convex in T_7.

Proof: The partial derivatives of the first and second order of $TC_F(A,T_7,s)$ with respect to T_7 are presented below:

$$\frac{\partial TC_F(A,T_7,s)}{\partial T_7} = \begin{bmatrix} -(k_r + k_s)\left(\frac{1}{T_7^2}\right) - Amk_a\left(\frac{1}{T_7^2}\right) + 3C_1(r - ls)A^\eta C_5^2\,T_7^2 \\ + A^\eta C_2(r - ls)(1 - 2C_5\mathrm{T}_7) + 3\,C_3(r - ls)^2A^{2\eta}C_5^2T_7^2 - 3C_4(r - ls)^3A^{3\eta}C_5^2T_7^2 \end{bmatrix}$$

The value of T_7 is obtained from the equation $\frac{\partial TC_F(A,T_5,T_7,s)}{\partial T_7} = 0$ and it is optimum if $\frac{\partial^2 TC_F(A,T_7,s)}{\partial {T_7}^2} > 0$. That is.,

$$\frac{\partial^2 TC_F(A,T_7,s)}{\partial {T_7}^2} = \begin{bmatrix} 2(k_r + k_s)\left(\frac{1}{T_7^3}\right) + 2Amk_a\left(\frac{1}{T_7^3}\right) + 6C_1\left((r - ls)A^\eta C_5^2T_7\right) \\ -2C_5\,C_2(r - ls)A^\eta + 6\,C_3(r - ls)^2A^{2\eta}C_5^2T_7 - 6C_4(r - ls)^3A^{3\eta}C_5^2T_7 \end{bmatrix} > 0$$

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This fulfills the proof of Theorem 3.3.

Theorem 3.4
There exists unique s^* that minimizes the total inventory cost function $TC_F(A, T_7, s)$ for a fixed value of A.
Proof: The partial derivatives of the first and second order of the total inventory cost function $TC_F(A, T_7, s)$ with respect to s are presented below:

$$\frac{\partial TC_F(A,T_7,s)}{\partial s} = \begin{bmatrix} -lk_pA^{\eta} - C_1lA^{\eta}C_5^2\,T_7^3 - \,C_2lA^{\eta}\left(T_7 - C_5T_7^2\right) \\ -2l\,C_3(r\,-\,ls)A^{2\eta}C_5^2T_7^3 + 3lC_4(r\,-\,ls)^2A^{3\eta}C_5^2T_7^3 \end{bmatrix}$$

The value of T_5 is obtained from the equation $\frac{\partial TC_F(A,T_5,T_7,s)}{\partial s} = 0$ and it is optimum if $\frac{\partial^2 TC_F(A,T_7,s)}{\partial s^2} > 0$. That is.,

$$\frac{\partial^2 TC_F(A,T_5,T_7,s)}{\partial s^2} = 2l^2\,C_3A^{2\eta}C_5^2T_7^3 - 6l^2C_4(r\,-\,ls)A^{3\eta}C_5^2T_7^3 > 0$$

This fulfills the proof of Theorem 3.4.
Since the equations $\frac{\partial TC_F(A,T_5,T_7,s)}{\partial A} = 0$, $\frac{\partial TC_F(A,T_5,T_7,s)}{\partial T_7} = 0$ and $\frac{\partial TC_F(A,T_5,T_7,s)}{\partial s} = 0$ are a nonlinear equations, it is extremely convoluted to find the value of A, T_7 and s. These states that the optimal solution could not be guaranteed. Nevertheless, with the help of simple search method such as Newton's or Bisection method, one can able to find the value of A, T_7 and s. Mathematica 9.0 software is used to obtain these parameter values.

4 Numerical example and sensitivity analysis

In the present section, a numerical example is provided to illustrate the developed inventory model. Sensitivity analysis is also given in order to provide the managerial insights of the inventory model.

4.1 Numerical example

4.1.1 Example problem
The present research gives the following numerical example to examine the theoretical results as discussed in section 3 with the parameter value of the inventory model: Assume that a production inventory system has three different production rates with the demand function taken by $D(A, s) = (7\,-\,2s)A^{0.001}$ where the market selling price is \$13 per unit. The vendor has to spend \$30 for a production setup per cycle, \$10 for rework process setup per cycle. The deterioration rate is 0.01/unit/cycle. The vendor has to maintain the product in the stock which cost him \$15/unit item/cycle. For advertisement, the vendor has to spend \$3/per setup/cycle. Let the fraction of producing serviceable items is 0.6 and the remaining part is imperfect which can be remanufactured. The rework process begins when production ends, at a rate 610/units/cycle. The supplementary parameters are reviewed as follows:
$\mathrm{a} = 1.4,\ \mathrm{b} = 2.1, \mathrm{k_a} = 0.7,\ \mathrm{h_r} = 195,\ \lambda = 1.6,\ \mathrm{D_c} = 10,\ \mathrm{c_s} = 15,\ \mathrm{u} = 0.2,\ \mathrm{v} = 0.7,\ \mathrm{w} = 0.9$.

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4.1.2 Solution

This problem is solved by using the Mathematica 9.0 software. The optimal inventory total cost is given by:

$\mathrm{TC_F} = \$\, 5853.53$ when $\mathrm{T}_7^* = 5.0750$ and $\mathrm{m}^* = 2.$

The optimal replenishment-run time $T_1^*, T_2^*, T_3^*, T_4^*, T_5^*$ and T_6^* are respectively given by:

$T_1^* = 0.0599$, $T_2^* = 0.0857$, $T_3^* = 0.1225$, $T_4^* = 1.9991$, $T_5^* = 3.3661$, $T_6^* = 4.6580.$

The optimal length of the planning time horizon is given by:

$H^* = 10.1500$ years

The optimal production-run time $\left(T_p^*\right)$ and the optimal production lot size quantity of a production cycle are respectively shown by:

$T_p^* = T_1^* + T_2^* + T_3^* = 0.2681$ years and $\quad Q^* = pT_p^* \approx 5933$ units

From equations (17), (19) and (21), the maximum production lot size quantity during the production-run time T_1, T_2 and T_3 are respectively given by:

$Q_1 \approx 370$ units $\quad Q_2 \approx 3541$ units $\quad Q_3 \approx 4598$ units

It is derived that the vendor produces 370 units of products in 0.0599 year, 3541 units of products in 0.0857 year and 4598 units of products in 0.1225 years to send consumers and marketplaces.

From equation (22), the maximum level of inventory of serviceable products is

$I_{ms} \approx 2237$ units.

That is, after the rework process ends, the vendor has 2237 units of serviceable products on hand, which are utilized to sell in the marketplaces during non-production-run time periods.

From equation (23), the amount of shortage quantities per production cycle is given by:

$S \approx 551$ units

As a result, the retailer has to get shortage of 551 units per production cycle.

From equations (35) and (36), the maximum level inventory of serviceable products during the production-run time T_1 and T_2 are respectively given by:

$R_1 \approx 59$ units $\quad R_2 \approx 174$ units

From equation (38), the maximum level of inventory of imperfect production is

$I_{mr} \approx 349$ units.

At the beginning time of the rework process, the vendor has 349 units of imperfect items on hand per production cycle.

5 Sensitivity analysis

In the section, a sensitivity analysis is carried out based on the numerical example discussed in the section (4.1). This analysis is carried out by varying one of the parameters at a time keeping the other parameters as constant.

Based on the numerical outcome (Table 2), the following observations could be found:

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Table 2 Sensitivity analysis for different parameter's value

Parameters	Changes	T_4^*	T_5^*	T_6^*	T_7^*	Q^*	S^*	$TC_F{}^*$
A	1	1.8732	3.5886	4.5874	5.0701	5645	188	5123.73
	2	1.9375	3.5397	4.6324	5.0724	5833	390	5562.51
	4	2.1074	3.4964	4.6954	5.0894	6003	532	5967.58
	5	2.784	3.4759	5.2364	5.0954	6235	605	6328.11
θ	0.02	1.9991	3.3661	4.6580	5.0750	5933	551	5453.63
	0.03	1.9991	3.3661	4.6580	5.0750	5933	551	5568.42
	0.04	1.9991	3.3661	4.6580	5.0750	5933	551	5952.14
	0.05	1.9991	3.3661	4.6580	5.0750	5933	551	6423.82
p_r	590	2.0521	3.2333	4.6580	5.0750	5933	551	6879.49
	600	2.0026	3.3845	4.6580	5.0750	5933	551	6124.48
	620	1.8865	3.6461	4.6580	5.0750	5933	551	5127.49
	630	1.8254	3.7750	4.9546	5.0750	5998	496	4982.10
D_c	8	1.9991	3.5325	4.6580	5.0650	5933	496	5468.17
	9	1.9991	3.5367	4.6580	5.0650	5933	496	5518.82
	11	1.9996	3.5189	4.8791	0.0700	6012	456	6547.49
	12	2.1123	3.4961	4.9874	5.0850	6125	489	7489.28
C_s	5	1.9991	3.3661	4.6580	5.0650	5933	490	5387.14
	10	1.9991	3.3661	4.6580	5.0650	5933	490	5487.29
	20	2.1480	3.5158	4.7542	5.0750	5987	460	6587.18
	25	2.5641	3.4920	4.8546	5.0850	6112	589	7845.29

5.1 Observations

This inventory model is extremely sensitive to the holding cost the imperfect products compared to the cost of holding good quality products due to the imperfect products are stocked for rework till the production-run time ends. If the fraction of the recoverable products raises, the total inventory average cost reduces. So, the company managers have to recover the defective items more. Increase the shortage cost leads to reduce in total cost. Nevertheless, it is observed that this inventory model is less sensitive to the cost of production setup and rework setup. If the production and rework setup costs increase, the total inventory cost increases up to certain limits and after that it remains as constant. The number of determined production cycle increases a smaller amount of total cost of the model also increases and the substantial amount of the cost of carrying products reduces. Hence, if the producer increases the number of the production cycle, a considerable sum of holding costs could be decreased.

5.2 Managerial insights

This inventory model is proposed for sensible problems focused on the production-oriented industries like deterioration, imperfect, reworking, shortage and screening. This inventory model is an addition to the research line. The idea applied at this point provides the company managers with more useful alternative for the reason that it prevents the imperfect items by the screening method and provides more precisely the economic production lot size, production-run time, number of defective units. This inventory model highlighted that the inspection process in this model is appropriate for company manufacturer who have an

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automated screening set up to avoid the defective and deteriorating products. This inventory model assists the industry managers to manufacture high quality items through eliminating imperfect products by 100% perfect screening method so that high quality products could be sold in the market and customers, which will make an optimistic impact on a company's icon.

6 Conclusion and future research

In this research, a deficient economic production quantity inventory model has been presented for imperfect products, in which there are three various levels of production and one rework setup. The proposed inventory model is suitable for newly launched products under steady pattern to a degree in time. Such situation of affairs has enthralled because from starting at a low production rate, huge quantum of stock of producing items at the unique stage could be avoided which will be led to a decrease in the cost of holding. In the inventory model, the products are categorized into two different groups which are good quality products and imperfect products. The imperfect items are remanufactured as serviceable products. The sensitivity analysis confirms that the $(M/P_3/R_1)$ policy decreases the total cost. The presented inventory model may help the producers and trader in determining the optimum production lot size, production-run time, length of production cycle time, number of replenishment cycles, number of units of imperfect products and total cost properly. This model also aids the producers to find out the number of units of defective items is manufactured per cycle time, in which the quantity of recoverable products could be found. For future research, the presented inventory model could be extended in many ways involving various rates of demand, like linear, quadratic and cubic demand, and Weibull deterioration rate with two-parameters. In addition, improper inspection of ensure quality (i.e., an examiner may categorize a serviceable item to be imperfect (Type I error) and he may also categorize a defective product to be a good quality one (Type II error)) will be taken into consideration.

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